A Characterization of Weighted Approximations by the Post-Widder and the Gamma Operators

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Abstract

We present a characterization of the approximation errors of the Post-Widder and the Gamma operators in $L_p(0,\infty)$, $1 \le p \le \infty$, with a weight x^γ for any real γ. Two types of characteristics are used – weighted K-functionals of the approximated function itself and the classical fixed step moduli of smoothness taken on a simple modification of it.

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1 Introduction

The Post-Widder operator is given by

$$P_s(f,x) = \frac{1}{\Gamma(s)} \int_0^\infty f\left(\frac{xv}{s}\right) e^{-v} v^s \frac{dv}{v}, \tag{1.1}$$

where f is a measurable function defined on $(0,\infty)$, Γ denotes as usual the Gamma function and s is a positive real parameter. This operator for integer s is actually the Post-Widder real inversion formula for the Laplace transform.

The Gamma operator, introduced by A. Lupas and M. Müller [9], is given by

$$G_s(f,x) = \frac{1}{\Gamma(s+1)} \int_0^\infty f\left(\frac{xs}{v}\right) e^{-v} v^{s+1} \frac{dv}{v}. \tag{1.2}$$

The two operators are closely related. If for real α we denote the power function by $\chi^\alpha(x) = x^\alpha$ for $x > 0$ and set $\tau_s(u) = \frac{s+1}{s}u$, then

$$G_s(f,x) = P_{s+1}(f \circ \chi^{-1} \circ \tau_s, \chi^{-1}(x)). \tag{1.3}$$

Both operators have a simple action on the power functions. Direct application of the definition of the Gamma function gives

$$P_s(\chi^\alpha) = \frac{\Gamma(s+\alpha)}{s^\alpha \Gamma(s)} \chi^\alpha, \quad G_s(\chi^\alpha) = \frac{s^\alpha \Gamma(s+1-\alpha)}{\Gamma(s+1)} \chi^\alpha. \tag{1.4}$$

These formulae show that the two operators preserve the functions $\chi^0(x) = 1$ and $\chi^1(x) = x$.

Both operators were extensively studied. Here we only discuss results on characterizing their rate of convergence in terms of proper K-functionals. In view of (1.3) all results formulated below for one of the operators can easily be proved for the other too.

For $r \in \mathbb{N}$, $1 \le p \le \infty$, $\gamma \in \mathbb{R}$, $D = \frac{d}{dx}$ and $\varphi = \chi$ we consider the weighted K-functionals:

$$\begin{aligned} K^r_\gamma(f, t^r)_p &= K(f, t^r; L_p(\chi^\gamma)(0,\infty), AC^{r-1}_{loc}, \varphi^r D^r) \\ &= \inf\left\{\|\chi^\gamma(f-g)\|_p + t^r\|\chi^\gamma\varphi^r D^r g\|_p : g \in AC^{r-1}_{loc}(0,\infty)\right\}, \quad (1.5) \end{aligned}$$

defined for every $f \in \pi_{r-1} + L_p(\chi^\gamma)(0,\infty)$ and $t > 0$. By π_k we denote the space of all algebraic polynomials of degree k. $AC^k_{loc}(a,b) = \{g : g, g', \dots, g^{(k)} \in AC[\bar{a},\bar{b}]\ \forall a < \bar{a} < \bar{b} < b\}$ and $AC[\bar{a},\bar{b}]$ is the set of the absolutely continuous functions on $[\bar{a},\bar{b}]$. Above and in what follows $L_\infty(\chi^\gamma)(0,\infty)$ can be replaced by the spaces $C(\chi^\gamma)(0,\infty) = \{f : \chi^\gamma f \in C(0,\infty)\}$, where $C(a,b)$ is the space of all continuous functions **bounded** on (a,b). When in (1.5) $g \in AC^{r-1}_{loc}$ is such that either $f - g \notin L_p(\chi^\gamma)$ or $D^r g \notin L_p(\chi^\gamma\varphi^r)$ we assume that $\|\chi^\gamma(f-g)\|_p + t^r\|\chi^\gamma\varphi^r D^r g\|_p = +\infty$.

Note that the weight in the second term in the right-hand side of (1.5) is $\chi^{\gamma+r}$. We use two notations (φ and χ) for one and the same function in order to underline the different role of the two multipliers in the discussions in Sections 2 and 3.

The direct theorem for the approximation error of the Gamma operator in L_p, $1 \le p \le \infty$, without weights is proved by Totik [12]:

$$\|f - G_s(f)\|_p \le cK^2_0(f, s^{-1})_p.$$

In the same article [12] a weak converse theorem of the form

$$K^2_0(f, s^{-1})_p \le cs^{-1}\left(\sum_{k=2}^{s}\|f - G_k(f)\|_p + \|f\|_p\right)$$

is obtained. Here and in the sequel we denote by c positive numbers independent of the functions f, the parameter t below and the parameter s of the operators. The numbers c may differ at each occurrence.

The book of Ditzian and Totik [3] extends the above direct result to weights equivalent to $w(x) = x^{\gamma_0}(1+x)^{\gamma_\infty}$ with arbitrary real exponents γ_0, γ_∞. The converse result for the same weights is given as a statement for the equivalent rates of convergence in terms of weighted Ditzian-Totik moduli.

The question for the validity of strong converse theorems (in the terminology of [2]) complementing the direct estimates remained open for a while. In 2002 Sangüesa [11] proved the strong converse theorem of type A for $\gamma = 0$, $p = \infty$, namely

$$K^2_0(f, s^{-1})_\infty \le c\|f - P_s(f)\|_\infty.$$

As far as we know this is the only strong converse theorem of type A for the Post-Widder or the Gamma operators proved by now. As for strong converse theorems of type B, two results have recently been published. In [7] Guo, Liu, Qi and Zhang proved that for $\gamma = 0$ and $1 \le p \le \infty$ there is a constant $m > 1$ such that

$$K_0^2(f, n^{-1})_p \le c\left(\|f - G_n(f)\|_p + \|f - G_{mn}(f)\|_p\right).$$

The other result is a similar strong converse theorem of type B, proved by Qi and Guo in [10] for $-2 \le \gamma \le 0$ and $p = \infty$.

One of the main results of this article is a strong converse theorem of type A for the Post-Widder and the Gamma operator for $\gamma \in \mathbb{R}$ and $1 \le p \le \infty$.

Theorem 1.1. *There are positive numbers N, M such that for every $\gamma \in \mathbb{R}$, $s \ge N(\gamma^2 + 1)$, $1 \le p \le \infty$ and $f \in \pi_1 + L_p(\chi^\gamma)(0, \infty)$ we have*

$$\|\chi^\gamma (f - P_s(f))\|_p \le \left(2 + M\frac{\gamma^2 + 1}{s}\right) K_\gamma^2(f, (4s)^{-1})_p \tag{1.6}$$

and

$$K_\gamma^2(f, (4s)^{-1})_p \le \left(\kappa + M\frac{1}{\sqrt{s}} + M\frac{\gamma^2 + 1}{s}\right) \|\chi^\gamma (f - P_s(f))\|_p \tag{1.7}$$

with

$$\kappa = \frac{21 - 4\sqrt{2}}{8 - 2\sqrt{2}} = 2.966824...$$

The same inequalities are true if P_s is replaced by G_s.

Inequalities like (1.6) are well-known. For example, they are proved in [12] and [3], but with bigger constants. The inverse inequality (1.7) seems to be new (except $\gamma = 0,\ p = \infty$). It comes with a very small constant κ. Thus, the ratio $\|\chi^\gamma (f - P_s(f))\|_p / K_\gamma^2(f, (4s)^{-1})_p$ is bounded between two numbers with ratio less than 6 when s is big enough!

Theorem 1.1 remains true (up to the value of the constants) if the weight χ^γ is replaced by any equivalent on $(0, \infty)$ weight.

The K-functional (1.5) is characterized in [3, Chapter 6] by the weighted Ditzian-Totik moduli of smoothness. But it turns out that $K_\gamma^r(f, t^r)_p$ has a simple characterization in terms of the classical (unweighted fixed-step) moduli of smoothness $\omega_k(F, t)_{p(\mathbb{R})}$. Following the ideas of [5] we obtain

Theorem 1.2. *Let $r \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, $0 < t \le t_0$ and $f \in L_p(\chi^{\alpha - 1/p})(0, \infty)$.*

a) If $\alpha \ne 1 - r, 2 - r, \ldots, -1, 0$, then

$$K_{\alpha - 1/p}^r(f, t^r)_p \sim \omega_r((\chi^\alpha f) \circ \mathcal{E}, t)_{p(\mathbb{R})} + t^r \|(\chi^\alpha f) \circ \mathcal{E}\|_{p(\mathbb{R})}.$$

b) If $\alpha = 1-r, 2-r, \ldots, -1, 0$, *then*

$$K^r_{\alpha-1/p}(f,t^r)_p \sim \omega_r((\chi^\alpha f)\circ \mathcal{E}, t)_{p(\mathbb{R})} + t^{r-1}\omega_1((\chi^\alpha f)\circ \mathcal{E}, t)_{p(\mathbb{R})}.$$

By $\mathcal{E}$ and $\mathcal{E}^\alpha$ we denote the exponential function and its powers, i.e. $\mathcal{E}(x) = e^x, \mathcal{E}^\alpha(x) = e^{\alpha x}, \alpha \in \mathbb{R}$. By $\Psi(f,t) \sim \Theta(f,t)$ we mean that there exists c such that $c^{-1}\Theta(f,t) \le \Psi(f,t) \le c\Theta(f,t)$ for all f and t under consideration.

The assertions of Theorem 1.2 follow from Theorems 6.6 and 7.3 proved below. Let us mention that Theorem 6.6 improves the result of [4, Theorem 1 with $\theta = \mathcal{E}$].

Remark 1.3. The characterization of $K^r_{\alpha-1/p}(f,t^r)_p$ splits into two types, which cannot be unified. Indeed, let $\psi \in C^r(\mathbb{R})$, $\psi \not\equiv 0$, be with a finite support. Set $F_n(x) = \psi(n^{-1}x)$, $n \in \mathbb{N}$. Then $\omega_k(F_n,t)_{p(\mathbb{R})} \sim n^{-k+1/p}\, t^k$ and

$$\omega_r(F_n,t)_{p(\mathbb{R})} + t^{r-k}\omega_k(F_n,t)_{p(\mathbb{R})} \sim n^{-k+1/p}t^r, \quad k = 0,1,\ldots,r,$$

where $\omega_0(F,t)_{p(\mathbb{R})}$ means $\|F\|_{p(\mathbb{R})}$. Hence, any two of the above quantities are not equivalent with constants independent of n and $t \in (0,1]$. See also Corollary 5.3.

From Theorem 1.1 and Theorem 1.2 we immediately get

Theorem 1.4. *Let* $\gamma \in \mathbb{R}$, $1 \le p \le \infty$, $f \in L_p(\chi^\gamma)(0,\infty)$ *and* $s \ge N(\gamma^2+1)$, *where* N *is from Theorem 1.1.*

a) If $\gamma \ne -1-1/p, -1/p$, *then*

$$\begin{aligned}\|\chi^\gamma(f-P_s(f))\|_{p(0,\infty)} &\sim \|\chi^\gamma(f-G_s(f))\|_{p(0,\infty)}\\ &\sim \omega_2((\chi^{\gamma+1/p}f)\circ\mathcal{E}, s^{-1/2})_{p(\mathbb{R})} + s^{-1}\,\|(\chi^{\gamma+1/p}f)\circ\mathcal{E}\|_{p(\mathbb{R})}.\end{aligned}$$

b) If $\gamma = -1-1/p, -1/p$, *then*

$$\begin{aligned}\|\chi^\gamma(f-P_s(f))\|_{p(0,\infty)} &\sim \|\chi^\gamma(f-G_s(f))\|_{p(0,\infty)}\\ &\sim \omega_2((\chi^{\gamma+1/p}f)\circ\mathcal{E}, s^{-1/2})_{p(\mathbb{R})} + s^{-1/2}\omega_1((\chi^{\gamma+1/p}f)\circ\mathcal{E}, s^{-1/2})_{p(\mathbb{R})}.\end{aligned}$$

In particular, for the case $\gamma = 0$, $p = \infty$ we obtain

$$\|f-P_s(f)\|_{\infty(0,\infty)} \sim \omega_2(f\circ\mathcal{E}, s^{-1/2})_{\infty(\mathbb{R})} + s^{-1/2}\omega_1(f\circ\mathcal{E}, s^{-1/2})_{\infty(\mathbb{R})}.$$

Remark 1.5. If $f \in \pi_1 + L_p(\chi^\gamma)(0,\infty)$ as in Theorem 1.1, then in the characterization of the errors above f is to be replaced by f_0 such that $f_0 \in L_p(\chi^\gamma)(0,\infty)$ and $f - f_0 \in \pi_1$.

The results of this paper have been announced in [6].

The paper is organized as follows. Section 2 contains the inequalities on which the proof of Theorem 1.1 is based. In Section 3 we give the proof of this theorem. Next, Section 4 is devoted to imbedding inequalities needed in the proof of the characterization of the K-functional $K^r_\gamma(f,t^r)_p$ by the classical moduli of smoothness. In Section 5 we give several auxiliary results on K-functionals. The proof of Theorem 1.2 naturally splits into two parts. In Section 6 we characterize $K^r_\gamma(f,t^r)_p$ by K-functionals on the real line with exponential weights taken on a modification of the function. In Section 7 we proceed further to estimate this weighted K-functionals by the classical moduli of smoothness by modifying the function again.

2 Inequalities for the Post-Widder operator

For $\beta\in\mathbb{R}$ and $s>\max\{0,\beta\}$ we set

$$\kappa_1(\beta,s):=\frac{s^\beta\Gamma(s-\beta)}{\Gamma(s)};$$

$$\kappa_j(\beta,s):=\frac{s^{j-1}}{(2j-3)!\Gamma(s)}\int_0^\infty\int_1^{v/s}\left(\frac{v}{sy}-1\right)^{2j-3}y^{-\beta}\frac{dy}{y}e^{-v}v^s\frac{dv}{v},\quad j=2,3,4;$$

$$\lambda_1(\beta,s):=\frac{s^{\beta-1}}{\Gamma(s)}\int_0^\infty|(v-s-1)^2-s-1|e^{-v}v^{s-\beta}\frac{dv}{v};$$

$$\lambda_2(\beta,s):=\frac{s^{\beta-1}}{\Gamma(s)}\int_0^\infty|(v-s-3)^2-s-3|e^{-v}v^{s-\beta}\frac{dv}{v};$$

$$\lambda_3(\beta,s):=\frac{s^{\beta-\frac12}}{\Gamma(s)}\int_0^\infty|v-s-2|e^{-v}v^{s-\beta}\frac{dv}{v};$$

The quantities $\kappa_j(\beta,s),\lambda_j(\beta,s)$ will be used in the inequalities established in Propositions 2.4 – 2.9. It is important for us that they remain bounded by absolute constants for $\beta\in\mathbb{R}$ and $s\ge\beta^2+8$.

Note that the signs of $(\frac{v}{sy}-1)^{2j-3}$ and $(\frac{v}{s}-1)$ in the definition of κ_j coincide for every y from the integration range. Hence, the inner integral is always a non-negative number. This fact will be used in Propositions 2.5 and 2.6.

Lemma 2.1. *For $\beta\in\mathbb{R}$ and $s>\max\{0,\beta\}$ we have*

$$\kappa_1(\beta,s)-1=\beta(\beta+1)\kappa_2(\beta,s)s^{-1};\tag{2.1}$$

$$\kappa_2(\beta,s)-\frac12=\left[(\beta+2)(\beta+3)\kappa_3(\beta,s)-\frac{\beta+2}{3}\right]s^{-1};\tag{2.2}$$

$$\kappa_3(\beta,s)-\frac18=\left[(\beta+4)(\beta+5)\kappa_4(\beta,s)-\frac{2\beta+5}{12}\right]s^{-1}-\frac{\beta+4}{5}s^{-2}.\tag{2.3}$$

Proof. Applying twice integration by parts we get for $j\ge2$

$$\int_1^z\left(\frac{z}{y}-1\right)^{2j-3}y^{-\beta}\frac{dy}{y}=\frac{(z-1)^{2j-2}}{2j-2}-\frac{(\beta+2j-2)(z-1)^{2j-1}}{(2j-2)(2j-1)}$$
$$+\frac{(\beta+2j-2)(\beta+2j-1)}{(2j-2)(2j-1)}\int_1^z\left(\frac{z}{y}-1\right)^{2j-1}y^{-\beta}\frac{dy}{y}.$$

When we plug this formula with $z = v/s$ in the definition of κ_j we get

$$\kappa_j(\beta, s) = \frac{s^{j-1}}{(2j-2)!}T(2j-2, s) - \frac{(\beta+2j-2)s^{j-1}}{(2j-1)!}T(2j-1, s) + \frac{(\beta+2j-2)(\beta+2j-1)}{s}\kappa_{j+1}(\beta, s), \tag{2.4}$$

where

$$T(m, s) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\prod_{i=0}^{k-1}\left(1+\frac{i}{s}\right). \tag{2.5}$$

As usual the product is 1 for an upper bound, which is smaller than the lower bound. Direct calculations show that formulae (2.4) – (2.5) remain true for $j = 1$. From (2.5) we get

$$T(0, s) = 1,\ T(1, s) = 0,\ T(2, s) = s^{-1},\ T(3, s) = 2s^{-2},$$
$$T(4, s) = 3s^{-2}(1 + 2s^{-1}),\ T(5, s) = 4s^{-3}(5 + 6s^{-1}).$$

Now, applying (2.4) with $j = 1, 2$ and 3 we complete the proof. □

Lemma 2.2. *There exists an absolute constant M_1 such that for every $s \geq \beta^2+8$ and $\beta \in \mathbb{R}$ we have*

$$|\kappa_1(\beta, s) - 1| \leq M_1\frac{1+\beta^2}{s}; \tag{2.6}$$

$$\left|\kappa_2(\beta, s) - \frac{1}{2}\right| \leq M_1\frac{1+\beta^2}{s}; \tag{2.7}$$

$$\left|\kappa_3(\beta, s) - \frac{1}{8}\right| \leq M_1\frac{1+\beta^2}{s}. \tag{2.8}$$

Proof. In view of Lemma 2.1 it is enough to prove the existence of a constant M_2 such that

$$0 < \kappa_j(\beta, s) \leq M_2 \quad \forall\ j = 1, 2, 3, 4, \quad \beta \in \mathbb{R}, \quad s \geq \beta^2 + 8. \tag{2.9}$$

First, we shall prove (2.9) for $j = 1$, which, in turn, will be used when establishing (2.9) for the other j's. Note that (2.1) implies $0 < \kappa_1(\beta, s) < 1$ for $-1 < \beta < 0$, $\kappa_1(-1, s) = \kappa_1(0, s) = 1$ and $1 < \kappa_1(\beta, s)$ for $\beta < -1$ or $0 < \beta$. For $\beta < 0$ using

$$\kappa_1(\beta, s) = \left(1 - \frac{\beta+1}{s}\right)...\left(1 - \frac{\beta+m}{s}\right)\frac{s^{\beta+m}\Gamma(s-\beta-m)}{\Gamma(s)}$$

with $m = [-\beta]$ and $m = [-\beta] + 1$ we get

$$1 - \frac{1}{s} \leq 1 - \frac{[-\beta]+1+\beta}{s} \leq \kappa_1(\beta, s)\prod_{i=1}^{[-\beta]}\left(1 - \frac{\beta+i}{s}\right)^{-1} \leq 1. \tag{2.10}$$

Now the last inequality in (2.10) implies

$$\kappa_1(\beta,s) \le \prod_{i=1}^{[-\beta]} \left(1+\frac{-\beta-i}{s}\right) \le e^{\sum_{i=1}^{[-\beta]}(-\beta-i)s^{-1}} \le e^{\beta^2(2s)^{-1}} \le \sqrt{e},$$

which verifies (2.9) for $j=1$ and $\beta<0$. For $\beta \ge 0$ using

$$\kappa_1(\beta,s) = \left(1-\frac{\beta}{s}\right)^{-1} \dots \left(1-\frac{\beta-m+1}{s}\right)^{-1} \frac{s^{\beta-m}\Gamma(s-\beta+m)}{\Gamma(s)}$$

with $m=[\beta]$ and $m=[\beta]+1$ we get

$$1-\frac{1}{s} \le 1-\frac{\beta-[\beta]}{s} \le \kappa_1(\beta,s)\prod_{i=0}^{[\beta]}\left(1-\frac{\beta-i}{s}\right) \le 1. \tag{2.11}$$

Having in mind that $\frac{\beta}{s} \le \frac{\beta^2+8}{5s} \le \frac{1}{5}$ we see as in the first case that the last inequality in (2.11) implies (2.9) for $j=1$ and $\beta \ge 0$.

In order to prove (2.9) for $j=2,\ 3$ and 4 we estimate from above the inner integral in the definition of $\kappa_j(\beta,s)$. For $j \ge 1$ we have

$$\int_1^{v/s}\left(\frac{v}{sy}-1\right)^{2j-1} y^{-\beta}\frac{dy}{y} = \int_1^{v/s}\left(\frac{v}{s}-y\right)^{2j-1} y^{-\beta-2j}dy$$
$$\le \int_1^{v/s}\left(\frac{v}{s}-y\right)^{2j-1} dy[1+v^{-\beta-2j}s^{\beta+2j}] = \frac{1}{2j}\left(\frac{v}{s}-1\right)^{2j}[1+v^{-\beta-2j}s^{\beta+2j}].$$

Hence

$$\kappa_{j+1}(\beta,s) = \frac{s^j}{(2j-1)!\Gamma(s)}\int_0^\infty\int_1^{v/s}\left(\frac{v}{sy}-1\right)^{2j-1} y^{-\beta}\frac{dy}{y}e^{-v}v^s\frac{dv}{v}$$
$$\le \frac{s^j}{(2j)!\Gamma(s)}\int_0^\infty\left(\frac{v}{s}-1\right)^{2j}[1+v^{-\beta-2j}s^{\beta+2j}]e^{-v}v^s\frac{dv}{v}$$
$$= \frac{s^j}{(2j)!\Gamma(s)}\int_0^\infty\left(\frac{v}{s}-1\right)^{2j}e^{-v}v^s\frac{dv}{v} + \frac{s^{j+\beta}}{(2j)!\Gamma(s)}\int_0^\infty\left(1-\frac{s}{v}\right)^{2j}e^{-v}v^{s-\beta}\frac{dv}{v}$$
$$= \frac{s^j}{(2j)!}\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}\frac{s^{-k}\Gamma(s+k)}{\Gamma(s)}$$
$$+ \frac{s^{j+\beta+2j}}{(2j)!}\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}\frac{s^{-k}\Gamma(s-\beta-2j+k)}{\Gamma(s)}$$
$$= \frac{s^j}{(2j)!}\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}\prod_{i=0}^{k-1}\left(1+\frac{i}{s}\right)$$
$$+ \frac{s^\beta\Gamma(s-\beta)}{\Gamma(s)}\frac{s^{2j}\Gamma(s-\beta-2j)}{\Gamma(s-\beta)}\frac{s^j}{(2j)!}\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}\prod_{i=0}^{k-1}\left(1-\frac{\beta+2j}{s}+\frac{i}{s}\right).$$

Therefore

$$(2j)!\kappa_{j+1}(\beta,s) \le T_j(0,s) + \frac{s^\beta\Gamma(s-\beta)}{\Gamma(s)}\frac{s^{2j}\Gamma(s-\beta-2j)}{\Gamma(s-\beta)}T_j(\beta+2j,s), \quad (2.12)$$

where

$$T_j(b,s) := s^j\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}\prod_{i=0}^{k-1}\left(1-\frac{b}{s}+\frac{i}{s}\right).$$

Direct calculations for $j=1,2,3$ give

$$\begin{aligned}
&T_1(b,s) = 1+b(b-1)s^{-1};\\
&T_2(b,s) = 3+2(3-7b+3b^2)s^{-1}+(-6b+11b^2-6b^3+b^4)s^{-2};\\
&T_3(b,s) = 15+5(26-33b+9b^2)s^{-1}+(120-404b+375b^2-130b^3+15b^4)s^{-2}\\
&\qquad\qquad + (-120b+274b^2-225b^3+85b^4-15b^5+b^6)s^{-3}
\end{aligned}$$

and in particular

$$T_1(0,s)=1;\quad T_2(0,s)=3+6s^{-1};\quad T_3(0,s)=15+130s^{-1}+120s^{-2}.$$

Substituting in (2.12) the above values of $T_j(b,s)$ with $b=0$ and $b=\beta+2j$, using (2.9) with $j=1$ and the inequality

$$\frac{s^{2j}\Gamma(s-\beta-2j)}{\Gamma(s-\beta)} = \prod_{i=1}^{2j}\left(1-\frac{\beta+i}{s}\right)^{-1} \le M_3,$$

valid for $\left|\frac{\beta+i}{s}\right| \le \frac{|\beta|+6}{\beta^2+8} \le \frac{4}{5}$, we prove (2.9) for $j=2,\ 3,\ 4$ and complete the proof of the lemma. $\square$

Remark 2.3. Note that the lower and upper estimates in (2.10) and (2.11) imply directly (2.6).

Proposition 2.4. *For every $f\in L_p(\chi^\gamma)(0,\infty)$, $1\le p\le\infty$, and $s>\max\{0,\gamma+p^{-1}\}$ we have*

$$\|\chi^\gamma P_s(f)\|_p \le \kappa_1(\gamma+p^{-1},s)\|\chi^\gamma f\|_p, \quad (2.13)$$

where $\kappa_1(\beta,s)$ is estimated in (2.6) for $s\ge\beta^2+8$.

Proof. From (1.1) we get

$$x^\beta P_s(f,x) = \frac{s^\beta}{\Gamma(s)}\int_0^\infty \left[\left(\frac{xv}{s}\right)^\beta f\left(\frac{xv}{s}\right)\right]e^{-v}v^{s-\beta}\frac{dv}{v}.$$

Applying the generalized Minkowski inequality in this representation we get

$$
\begin{aligned}
&\left\{\int_0^\infty |x^\beta P_s(f,x)|^p \frac{dx}{x}\right\}^{\frac{1}{p}}\\
&\quad \le \frac{s^\beta}{\Gamma(s)}\int_0^\infty \left\{\int_0^\infty \left|\left(\frac{xv}{s}\right)^\beta f\left(\frac{xv}{s}\right)\right|^p \frac{dx}{x}\right\}^{\frac{1}{p}} e^{-v}v^{s-\beta}\frac{dv}{v}\\
&\quad = \frac{s^\beta}{\Gamma(s)}\int_0^\infty \left\{\int_0^\infty \left|y^\beta f(y)\right|^p \frac{dy}{y}\right\}^{\frac{1}{p}} e^{-v}v^{s-\beta}\frac{dv}{v}\\
&\quad = \kappa_1(\beta,s)\left\{\int_0^\infty \left|y^\beta f(y)\right|^p \frac{dy}{y}\right\}^{\frac{1}{p}}.
\end{aligned}
$$

Putting $\beta = \gamma + p^{-1}$ in the above inequality we prove (2.13). □

Proposition 2.5. *For every g such that $\varphi^2 D^2 g \in L_p(\chi^\gamma)(0,\infty)$, $1 \le p \le \infty$, and $s > \max\{0, \gamma + p^{-1}\}$ we have*

$$
\|\chi^\gamma\left(P_s(g) - g\right)\|_p \le s^{-1}\kappa_2(\gamma + p^{-1}, s)\|\chi^\gamma \varphi^2 D^2 g\|_p, \tag{2.14}
$$

where $\kappa_2(\beta, s)$ is estimated in (2.7) *for $s \ge \beta^2 + 8$.*

Proof. Applying P_s to the Taylor expansion of g

$$
g(y) = g(x) + (y-x)g'(x) + \int_x^y (y-u)g''(u)\,du
$$

we get in view of (1.4)

$$
\begin{aligned}
P_s(g,x) - g(x) &= \frac{1}{\Gamma(s)}\int_0^\infty \int_x^{xv/s}\left(\frac{xv}{s} - u\right) g''(u)\,du\, e^{-v}v^s\,\frac{dv}{v}\\
&= \frac{1}{\Gamma(s)}\int_0^\infty \int_1^{v/s}\left(\frac{v}{sy} - 1\right)(xy)^2 g''(xy)\,\frac{dy}{y}\, e^{-v}v^s\,\frac{dv}{v}
\end{aligned}
$$

and hence

$$
x^\beta|P_s(g,x) - g(x)| \le \frac{1}{\Gamma(s)}\int_0^\infty \int_1^{v/s}\left(\frac{v}{sy} - 1\right) y^{-\beta}(xy)^{\beta+2}|g''(xy)|\,\frac{dy}{y}\, e^{-v}v^s\,\frac{dv}{v}.
$$

Now we apply the arguments from the proof of Proposition 2.4 in order to get (2.14). □

Proposition 2.6. *For every g such that $\varphi^4 D^4 g \in L_p(\chi^\gamma)(0,\infty)$, $1 \le p \le \infty$, and $s > \max\{0, \gamma + p^{-1}\}$ we have*

$$
\begin{aligned}
&\left\|\chi^\gamma\left(P_s(g) - g - \frac{1}{2}s^{-1}\varphi^2 D^2 g - \frac{1}{3}s^{-2}\varphi^3 D^3 g\right)\right\|_p\\
&\qquad\qquad \le s^{-2}\kappa_3(\gamma + p^{-1}, s)\|\chi^\gamma \varphi^4 D^4 g\|_p,
\end{aligned}
\tag{2.15}
$$

where $\kappa_3(\beta, s)$ is estimated in (2.8) *for $s \ge \beta^2 + 8$.*

Proof. Applying P_s to the Taylor expansion of g

$$g(y) = g(x) + (y-x)g'(x) + \frac{(y-x)^2}{2}g''(x) + \frac{(y-x)^3}{6}g'''(x) + \int_x^y \frac{(y-u)^3}{6} D^4 g(u)\, du$$

we get as in the proof of Proposition 2.5

$$x^\beta \left| P_s(g,x) - g(x) - \frac{1}{2}s^{-1}\varphi^2(x)D^2g(x) - \frac{1}{3}s^{-2}\varphi^3(x)D^3g(x) \right| \leq \frac{1}{6\Gamma(s)} \int_0^\infty \int_1^{v/s} \left(\frac{v}{sy} - 1\right)^3 y^{-\beta}(xy)^{\beta+4}|D^4g(xy)|\, \frac{dy}{y}\, e^{-v}v^s\, \frac{dv}{v}.$$

Now we apply the arguments from the proof of Proposition 2.4 in order to get (2.15). □

Proposition 2.7. *For every $f \in L_p(\chi^\gamma)(0,\infty)$, $1 \leq p \leq \infty$, and $s > \max\{0, \gamma + p^{-1}\}$ we have*

$$\|\chi^\gamma \varphi^2 D^2 P_s(f)\|_p \leq s\lambda_1(\gamma + p^{-1}, s)\|\chi^\gamma f\|_p. \tag{2.16}$$

There is an absolute constant M_4 such that

$$\lambda_1(\beta, s) \leq \sqrt{2} + M_4(1+\beta^2)s^{-1} \tag{2.17}$$

for every $\beta \in \mathbb{R}, s \geq \beta^2 + 8$.

Proof. Substituting $v = su/x$ in (1.1) we get

$$P_s(f,x) = \frac{1}{\Gamma(s)} \int_0^\infty f(u)e^{-su/x}s^s u^s x^{-s}\frac{du}{u}.$$

Differentiating the above expression twice with respect to x and making the inverse substitution $u = xv/s$ we arrive at

$$D^2P_s(f,x) = \frac{x^{-2}}{\Gamma(s)} \int_0^\infty f\left(\frac{xv}{s}\right)\left[(v-s-1)^2 - s - 1\right] e^{-v}v^s \frac{dv}{v}.$$

Hence

$$x^{\beta+2}|D^2P_s(f,x)| \leq \frac{s^\beta}{\Gamma(s)} \int_0^\infty \left(\frac{xv}{s}\right)^\beta \left|f\left(\frac{xv}{s}\right)\right| \left|(v-s-1)^2 - s - 1\right| e^{-v}v^{s-\beta}\frac{dv}{v}.$$

Now we apply the arguments from the proof of Proposition 2.4 in order to get (2.16). The estimate of λ_1 uses standard arguments – the Cauchy-Schwarz inequality. We have

$$
\begin{aligned}
&s^{-\beta+1}\Gamma(s)\lambda_1(\beta,s)\\
&\le \left\{\int_0^\infty \left((v-s-1)^2-s-1\right)^2 e^{-v}v^{s-\beta}\frac{dv}{v}\right\}^{1/2}\left\{\int_0^\infty e^{-v}v^{s-\beta}\frac{dv}{v}\right\}^{1/2}\\
&=\{\Gamma(s-\beta+4)-4(s+1)\Gamma(s-\beta+3)+2(s+1)(3s+2)\Gamma(s-\beta+2)\\
&-4s(s+1)^2\Gamma(s-\beta+1)+s^2(s+1)^2\Gamma(s-\beta)\}^{1/2}\Gamma(s-\beta)^{1/2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1(\beta,s) &\le \frac{s^\beta\Gamma(s-\beta)}{\Gamma(s)}\left\{2+\frac{2+4\beta(\beta-1)}{s}+\frac{\beta(\beta-1)(\beta^2-\beta+2)}{s^2}\right\}^{1/2}\\
&\le \sqrt{2}+M_4(1+\beta^2)s^{-1}.
\end{aligned}
$$

This proves (2.17). □

Proposition 2.8. *For every g such that $\varphi^2D^2g\in L_p(\chi^\gamma)(0,\infty)$, $1\le p\le\infty$, and $s>\max\{0,\gamma+p^{-1}\}$ we have*

$$
\|\chi^\gamma\varphi^4D^4P_s(g)\|_p \le s\lambda_2(\gamma+p^{-1},s)\|\chi^\gamma\varphi^2D^2g\|_p. \tag{2.18}
$$

There is an absolute constant M_5 such that

$$
\lambda_2(\beta,s)\le\sqrt{2}+M_5(1+\beta^2)s^{-1} \tag{2.19}
$$

for every $\beta\in\mathbb{R}$, $s\ge\beta^2+8$.

Proof. Differentiating (1.1) twice with respect to x, substituting $v=su/x$ in the right-hand side integral, differentiating the resulting expression twice with respect to x and making the inverse substitution $u=xv/s$ we arrive at

$$
D^4P_s(g,x)=\frac{x^{-4}}{\Gamma(s)}\int_0^\infty\left(\frac{xv}{s}\right)^2D^2g\left(\frac{xv}{s}\right)\left[(v-s-3)^2-s-3\right]e^{-v}v^s\frac{dv}{v}.
$$

Hence

$$
\begin{aligned}
&x^{\beta+4}|D^4P_s(g,x)|\\
&\qquad\le\frac{s^\beta}{\Gamma(s)}\int_0^\infty\left(\frac{xv}{s}\right)^{\beta+2}\left|D^2g\left(\frac{xv}{s}\right)\right|\left|(v-s-3)^2-s-3\right|e^{-v}v^{s-\beta}\frac{dv}{v}.
\end{aligned}
$$

Now we apply the arguments from the proof of Proposition 2.4 in order to get (2.18). As in the proof of Proposition 2.7 we estimate λ_2 by

$$
\begin{aligned}
\lambda_2(\beta,s) &\le \frac{s^\beta\Gamma(s-\beta)}{\Gamma(s)}\left\{2+\frac{18+4\beta(\beta+3)}{s}+\frac{36+\beta(\beta+3)(\beta^2+3\beta+14)}{s^2}\right\}^{1/2}\\
&\le\sqrt{2}+M_5(1+\beta^2)s^{-1}.
\end{aligned}
$$

This proves (2.19). □

Proposition 2.9. *For every g such that $\varphi^2 D^2 g \in L_p(\chi^\gamma)(0,\infty)$, $1 \le p \le \infty$, and $s > \max\{0, \gamma + p^{-1}\}$ we have*

$$\|\chi^\gamma \varphi^3 D^3 P_s(g)\|_p \le \sqrt{s}\lambda_3(\gamma + p^{-1}, s)\|\chi^\gamma \varphi^2 D^2 g\|_p, \tag{2.20}$$

There is an absolute constant M_6 such that

$$\lambda_3(\beta, s) \le 1 + M_6(1+\beta^2)s^{-1} \tag{2.21}$$

for every $\beta \in \mathbb{R}, s \ge \beta^2 + 8$.

Proof. Differentiating (1.1) twice with respect to x, substituting $v = su/x$ in the right-hand side integral, differentiating the resulting expression once with respect to x and making the inverse substitution $u = xv/s$ we arrive at

$$D^3 P_s(g,x) = \frac{x^{-3}}{\Gamma(s)} \int_0^\infty \left(\frac{xv}{s}\right)^2 D^2 g\left(\frac{xv}{s}\right)[v - s - 2]\, e^{-v} v^s \frac{dv}{v}.$$

Hence

$$x^{\beta+3}|D^3 P_s(g,x)| \le \frac{s^\beta}{\Gamma(s)} \int_0^\infty \left(\frac{xv}{s}\right)^{\beta+2} \left|D^2 g\left(\frac{xv}{s}\right)\right| |v - s - 2|\, e^{-v} v^{s-\beta} \frac{dv}{v}.$$

Now we apply the arguments from the proof of Proposition 2.4 in order to get (2.20). As in the proof of Proposition 2.7 we estimate λ_3 by

$$\lambda_3(\beta, s) \le \frac{s^\beta \Gamma(s-\beta)}{\Gamma(s)} \left\{1 + \frac{\beta^2 + 3\beta + 4}{s}\right\}^{1/2} \le 1 + M_6(1+\beta^2)s^{-1}.$$

This proves (2.21). ☐

Remark 2.10. The constant κ_1 in (2.13) of Proposition 2.4 is exact for $p = \infty$ as the example of $f_0(x) = x^{-\gamma}$ shows. The same example can be used to show that the constants κ_2 in (2.14) of Proposition 2.5 and κ_3 in (2.15) of Proposition 2.6 are exact for $p = \infty$ when $\gamma \ne 0, -1$ and $\gamma \ne 0, -1, -2, -3$ respectively. For the exceptional values of γ an additional logarithmic factor has to be introduced in the definition of the extremal function f_0. The constants are also exact for $1 \le p < \infty$. This can be seen if we multiply the extremal functions for $p = \infty$ with the characteristic function of the interval $[\varepsilon, \varepsilon^{-1}]$ and let $\varepsilon \to 0+$.

Remark 2.11. The constants λ_j in (2.16), (2.18) and (2.20) are not exact.

Remark 2.12. If the Post-Widder operator P_s is replaced by the Gamma operator G_s, then the results of this section remain true with slight changes. The necessary modifications are:

a) In Propositions 2.4 and 2.5 the restriction on s is $s > \max\{0, -\gamma - p^{-1} - 1\}$ and $\kappa_j(\gamma + p^{-1}, s)$ are replaced by $\kappa_j(-\gamma - p^{-1} - 1, s)$, $j = 1, 2$.

b) In Proposition 2.6 the restriction on s is $s > \max\{2, -\gamma - p^{-1} - 1\}$ and estimate (2.15) changes to

$$\left\| \chi^\gamma \left(G_s(g) - g - \frac{\varphi^2 D^2 g}{2(s-1)} - \frac{2\varphi^3 D^3 g}{3(s-1)(s-2)} \right) \right\|_p \leq \frac{\bar{\kappa}_3(\gamma + p^{-1}, s)}{s^2} \|\chi^\gamma \varphi^4 D^4 g\|_p,$$

where

$$\bar{\kappa}_3(\beta, s) := \frac{s^4}{6\Gamma(s)} \int_0^\infty \int_1^{v/s} \left(\frac{v}{sy} - 1 \right)^3 y^{\beta+3} \frac{dy}{y} e^{-v} v^{s-2} \frac{dv}{v}.$$

$\bar{\kappa}_3(\beta, s)$ satisfies (2.8) as κ_3 does.

c) In Proposition 2.7 the restriction on s is $s > \max\{0, -\gamma - p^{-1} - 1\}$ and $\lambda_1(\gamma + p^{-1}, s)$ is replaced by $\lambda_1(-\gamma - p^{-1} - 1, s)$.

d) In Proposition 2.8 the restriction on s is $s > \max\{0, -\gamma - p^{-1} - 1\}$ and $\lambda_2(\gamma + p^{-1}, s)$ is replaced by $\bar{\lambda}_2(-\gamma - p^{-1} - 1, s)$, where

$$\bar{\lambda}_2(\beta, s) := \frac{s^{\beta-1}}{\Gamma(s)} \int_0^\infty |(v - s + 1)^2 - s + 1| e^{-v} v^{s-\beta} \frac{dv}{v}.$$

$\bar{\lambda}_2(\beta, s)$ satisfies (2.19) as λ_2 does.

e) In Proposition 2.9 the restriction on s is $s > \max\{0, -\gamma - p^{-1} - 1\}$ and $\lambda_3(\gamma + p^{-1}, s)$ is replaced by $\bar{\lambda}_3(-\gamma - p^{-1} - 1, s)$, where

$$\bar{\lambda}_3(\beta, s) := \frac{s^{\beta-\frac{1}{2}}}{\Gamma(s)} \int_0^\infty |v - s + 1| e^{-v} v^{s-\beta} \frac{dv}{v}.$$

$\bar{\lambda}_3(\beta, s)$ satisfies (2.21) as λ_3 does.

3 A characterization of the Post-Widder operator error

Now we are ready to prove Theorem 1.1.

Proof. Both sides of (1.6) and (1.7) do not change if we subtract a linear function from f. So we may assume that $f \in L_p(\chi^\gamma)(0, \infty)$.

For every $g \in AC^1_{loc}(0, \infty)$ such that $g, \varphi^2 D^2 g \in L_p(\chi^\gamma)(0, \infty)$ we have from Propositions 2.4 and 2.5

$$\begin{aligned}
\|\chi^\gamma(P_s f - f)\|_p &\leq \|\chi^\gamma P_s(f - g)\|_p + \|\chi^\gamma(P_s g - g)\|_p + \|\chi^\gamma(f - g)\|_p \\
&\leq (\kappa_1 + 1)\|\chi^\gamma(f - g)\|_p + s^{-1}\kappa_2\|\chi^\gamma \varphi^2 D^2 g\|_p \\
&\leq \max\{\kappa_1 + 1, 4\kappa_2\}\{\|\chi^\gamma(f - g)\|_p + (4s)^{-1}\|\chi^\gamma \varphi^2 D^2 g\|_p\}.
\end{aligned}$$

(The arguments $\gamma + p^{-1}$ and s of κ_j, λ_j are omitted in the proof.) Taking infimum on g we get

$$\|\chi^\gamma(P_s f - f)\|_p \le \max\{\kappa_1 + 1, 4\kappa_2\} K_\gamma \left(f, \frac{1}{4s}\right)_p,$$

which, in view of (2.6), (2.7), proves (1.6).

In order to prove (1.7) for a given $f \in L_p(\chi^\gamma)(0,\infty)$ we set $g = P_s^2 f$. Then $\varphi^4 D^4 g \in L_p(\chi^\gamma)(0,\infty)$ in view of Propositions 2.7 and 2.8 (with $g = P_s f$) and hence we can apply Proposition 2.6. A consecutive application of Propositions 2.6, 2.8 and 2.7 gives

$$\begin{aligned}
&\left\|\chi^\gamma\left(P_s^3 f - P_s^2 f - \frac{1}{2s}\varphi^2 D^2 P_s^2 f - \frac{1}{3s^2}\varphi^3 D^3 P_s^2 f\right)\right\|_p \\
&\le \frac{\kappa_3}{s^2}\|\chi^\gamma \varphi^4 D^4 P_s^2 f\|_p \le \frac{\kappa_3\lambda_2}{s}\|\chi^\gamma\varphi^2 D^2 P_s f\|_p \\
&\le \frac{\kappa_3\lambda_2}{s}\|\chi^\gamma\varphi^2 D^2 P_s^2 f\|_p + \frac{\kappa_3\lambda_2}{s}\|\chi^\gamma\varphi^2 D^2 P_s(f - P_s f)\|_p \\
&\le \frac{\kappa_3\lambda_2}{s}\|\chi^\gamma\varphi^2 D^2 P_s^2 f\|_p + \kappa_3\lambda_2\lambda_1\|\chi^\gamma(f - P_s f)\|_p. \qquad (3.1)
\end{aligned}$$

Using Propositions 2.9 and 2.7 we obtain

$$\begin{aligned}
&\|\chi^\gamma\varphi^3 D^3 P_s^2 f\|_p \le s^{1/2}\lambda_3\|\chi^\gamma\varphi^2 D^2 P_s f\|_p \\
&\le s^{1/2}\lambda_3\|\chi^\gamma\varphi^2 D^2 P_s^2 f\|_p + s^{1/2}\lambda_3\|\chi^\gamma\varphi^2 D^2 P_s(f - P_s f)\|_p \\
&\le s^{1/2}\lambda_3\|\chi^\gamma\varphi^2 D^2 P_s^2 f\|_p + s^{3/2}\lambda_3\lambda_1\|\chi^\gamma(f - P_s f)\|_p. \qquad (3.2)
\end{aligned}$$

From (3.1), Proposition 2.4 and (3.2) we obtain

$$\begin{aligned}
&\frac{1}{2s}\|\chi^\gamma\varphi^2 D^2 P_s^2 f\|_p \\
&\le \left\|\chi^\gamma\left(P_s^3 f - P_s^2 f - \frac{1}{2s}\varphi^2 D^2 P_s^2 f - \frac{1}{3s^2}\varphi^3 D^3 P_s^2 f\right)\right\|_p \\
&+ \|\chi^\gamma P_s^2(P_s f - f)\|_p + \frac{1}{3s^2}\|\chi^\gamma\varphi^3 D^3 P_s^2 f\|_p \\
&\le \frac{\kappa_3\lambda_2}{s}\|\chi^\gamma\varphi^2 D^2 P_s^2 f\|_p + \kappa_3\lambda_1\lambda_2\|\chi^\gamma(f - P_s f)\|_p \\
&+ \kappa_1^2\|\chi^\gamma(f - P_s f)\|_p + \frac{\lambda_3}{3s^{3/2}}\|\chi^\gamma\varphi^2 D^2 P_s^2 f\|_p + \frac{\lambda_1\lambda_3}{3s^{1/2}}\|\chi^\gamma(f - P_s f)\|_p.
\end{aligned}$$

Hence

$$\frac{1}{4s}\|\chi^\gamma\varphi^2 D^2 P_s^2 f\|_p \le \frac{\kappa_1^2 + \kappa_3\lambda_1\lambda_2 + 1/3\lambda_1\lambda_3 s^{-1/2}}{2 - 4\kappa_3\lambda_2 - 4/3\lambda_3 s^{-1/2}}\|\chi^\gamma(f - P_s f)\|_p \qquad (3.3)$$

provided that $2 - 4\kappa_3\lambda_2 - 4/3\lambda_3 s^{-1/2} > 0$. This inequality is valid for $s \ge$

$N(\gamma^2+1)$ if we take into account (2.8), (2.19) and (2.21). Therefore

$$K_\gamma\left(f,\frac{1}{4s}\right)_p \le \|\chi^\gamma(f-P_s^2f)\|_p+\frac{1}{4s}\|\chi^\gamma\varphi^2D^2P_s^2f\|_p$$
$$\le\left(1+\kappa_1+\frac{\kappa_1^2+\kappa_3\lambda_1\lambda_2+1/3\lambda_1\lambda_3s^{-1/2}}{2-4\kappa_3\lambda_2-4/3\lambda_3s^{-1/2}}\right)\|\chi^\gamma(f-P_sf)\|_p.$$

In view of the estimates of κ_j and λ_j this inequality proves (1.7) and completes the proof of Theorem 1.1 for the Post-Widder operator P_s. The proof for the Gamma operator G_s is the same as we take into account Remark 2.12. □

In the proof of Theorem 1.1 (see (3.3) above) we have established the following statement which is of importance in itself.

Proposition 3.1. *There are positive numbers N,M such that for every $\gamma\in\mathbb{R}$, $s\ge N(\gamma^2+1)$, $1\le p\le\infty$ and $f\in\pi_1+L_p(\chi^\gamma)(0,\infty)$ we have*

$$\frac{1}{4s}\|\chi^\gamma\varphi^2D^2P_s^2f\|_p\le\left(\frac{5}{8-2\sqrt{2}}+M\frac{1}{\sqrt{s}}+M\frac{\gamma^2+1}{s}\right)\|\chi^\gamma(f-P_sf)\|_p.$$

Remark 3.2. The proof of the theorem follows an idea from [2]. The inequality $\kappa_3\lambda_2<\frac{1}{2}$ (here $\frac{1}{2}$ is the coefficient in front of $s^{-1}\varphi^2D^2g$ in the left-hand side of (2.15)) is crucial. The fact that the power -2 of s in front of φ^3D^3g in (2.15) is less than $-\frac{3}{2}$ is also of high importance. The values of the constants in the remaining propositions of Section 2 are not essential in this proof.

Remark 3.3. From Proposition 2.4, (3.3) and (1.6) we get

$$\|\chi^\gamma(f-P_s^2f)\|_p+\frac{1}{4s}\|\chi^\gamma\varphi^2D^2P_s^2f\|_p\le 2.98\|\chi^\gamma(f-P_sf)\|_p$$
$$\le 6K_\gamma\left(f,\frac{1}{4s}\right)_p \tag{3.4}$$

for s big enough. This means that P_s^2f provides a realization of the K-functional $K_\gamma(f,(4s)^{-1})_p$. The same is true for the other powers P_s^mf of the operator. For example, for $m=1$ from (3.3) and Proposition 2.7 we get

$$\frac{1}{2s}\|\chi^\gamma\varphi^2D^2P_sf\|_p\le\frac{1}{2s}\|\chi^\gamma\varphi^2D^2P_s^2f\|_p+\frac{1}{2s}\|\chi^\gamma\varphi^2D^2P_s(P_sf-f)\|_p$$
$$\le 2.7\|\chi^\gamma(f-P_sf)\|_p \tag{3.5}$$

for s big enough. Now (3.5) and (1.6) implies an inequality for P_sf similar to (3.4).

4 Imbedding inequalities

The proof of the characterization of the K-functional $K_\gamma^r(f,t^r)_p$ is based on several imbedding inequalities. As it is known for $g\in W_p^r[a,b]$ there holds

$$(b-a)^j\,\|g^{(j)}\|_{p[a,b]}\le c\left(\|g\|_{p[a,b]}+(b-a)^r\,\|g^{(r)}\|_{p[a,b]}\right),\quad j=0,1,\dots,r, \tag{4.1}$$

where the constant c depends only on r (see e.g. [1, p. 38]). As usual $W_p^r[a,b]$ denotes the space of the functions $g \in AC_{loc}^{r-1}[a,b]$ for which $f, f^{(r)} \in L_p[a,b]$. Using (4.1) one can show (cf. [4])

Proposition 4.1. *Let $r \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $1 \le p \le \infty$. Then for every $g \in AC_{loc}^{r-1}(0,\infty)$ such that $g, \chi^r g^{(r)} \in L_p(\chi^\gamma)(0,\infty)$ we have*

$$\|\chi^{\gamma+j} g^{(j)}\|_{p(0,\infty)} \le c\left(\|\chi^\gamma g\|_{p(0,\infty)} + \|\chi^{\gamma+r} g^{(r)}\|_{p(0,\infty)}\right), \quad j = 0,1,\dots,r, \tag{4.2}$$

where the constant c depends only on γ and r.

Proof. Using (4.1), we get for $a > 0$ and $j = 0,1,\dots,r$,

$$\begin{aligned}\|\chi^{\gamma+j} g^{(j)}\|_{p[a,2a]} &\le \max\{1, 2^{\gamma+j}\}\, a^{\gamma+j}\, \|g^{(j)}\|_{p[a,2a]} \\ &\le c\, a^\gamma \left(\|g\|_{p[a,2a]} + a^r\, \|g^{(r)}\|_{p[a,2a]}\right) \\ &\le c\left(\|\chi^\gamma g\|_{p[a,2a]} + \|\chi^{\gamma+r} g^{(r)}\|_{p[a,2a]}\right),\end{aligned} \tag{4.3}$$

where the constant c depends only on γ and r.

To prove (4.2) we divide the interval $(0,\infty)$ by the points $a_k = 2^k$, $k \in \mathbb{Z}$ and apply (4.3) on every interval $[a_k, a_{k+1}]$. Thus the case $p = \infty$ is settled. If $p < \infty$, we further raise both sides of (4.3) to power p, use the inequality $(A+B)^p \le 2^{p-1}(A^p + B^p)$, sum the inequalities in k and finally raise to power $1/p$. □

We derive the following corollary from Proposition 4.1, using the well-known Hardy's inequalities (see [8, p. 245]).

Corollary 4.2. *Let $r \in \mathbb{N}$, $i \in \{0,1,\dots,r-1\}$, $1 \le p \le \infty$ and $\gamma \in \mathbb{R}$ be such that $\gamma \ne 1-r-1/p, \dots, -i-1/p$. Then for $g \in AC_{loc}^{r-1}(0,\infty)$ such that $g, \chi^r g^{(r)} \in L_p(\chi^\gamma)(0,\infty)$ there hold*

$$\|\chi^{\gamma+j} g^{(j)}\|_{p(0,\infty)} \le c\, \|\chi^{\gamma+r} g^{(r)}\|_{p(0,\infty)}, \quad j = i, i+1, \dots, r-1, \tag{4.4}$$

where the constant c depends only on $\min\{|\gamma+j+1/p| : j = i, i+1, \dots, r-1\}$, *$\gamma$ and r.*

Proof. It is enough to prove the statement for $i = j = r-1$, since the general case follows from it by iteration. Since $g, \chi^r g^{(r)} \in L_p(\chi^\gamma)(0,\infty)$, then Proposition 4.1 yields that $\chi^{r-1} g^{(r-1)} \in L_p(\chi^\gamma)(0,\infty)$, i.e. $\chi^{\gamma+r-1} g^{(r-1)} \in L_p(0,\infty)$.

First, we consider the case $\gamma + r - 1 < -1/p$. From Hölder's inequality we get $\int_x^a \left|g^{(r)}(y)\right| dy \le c\, \|\chi^{\gamma+r} g^{(r)}\|_{p[0,a]}$ for $0 < x \le a$, which implies $g^{(r)} \in L_1[0,a]$. Moreover, the assumption $|g^{(r-1)}(x)| \ge c > 0$ in a neighborhood of the origin would imply $\chi^{\gamma+r-1} \in L_p[0,1]$, which contradicts $\gamma + r - 1 < -1/p$. Hence, there exists a sequence $\{\xi_n\}$ such that $\xi_n \to 0+0$ and $g^{(r-1)}(\xi_n) \to 0$ as $n \to \infty$. Combining these two facts with the representation $g^{(r-1)}(x) = g^{(r-1)}(\xi) + \int_\xi^x g^{(r)}(y)\, dy$, $0 < x, \xi \le a$ we get

$$g^{(r-1)}(x) = \int_0^x g^{(r)}(y)\, dy, \quad x \in (0,\infty), \tag{4.5}$$

and now Hardy's inequalities prove (4.4).

In a similar way in the case $\gamma+r-1>-1/p$ we show that the representation

$$g^{(r-1)}(x) = -\int_x^\infty g^{(r)}(y)\,dy, \quad x\in(0,\infty), \tag{4.6}$$

holds and once again Hardy's inequalities prove (4.4). □

Corollary 4.2 shows that, except for few values of γ, the conclusion of Proposition 4.1 can be improved by omitting $\|\chi^\gamma g\|_{p(0,\infty)}$ from the right-hand side of (4.2). Be aware that the condition $g\in L_p(\chi^\gamma)(0,\infty)$ is necessary for the validity of (4.4) as the example of $g(x)=x^j$ shows. Comparing Corollary 4.2 with [4, Lemma 3] we see that the conclusions are similar but the assumptions differ.

As a consequence of (4.5) and (4.6) we get the following simple description of the boundary behaviour of g.

Corollary 4.3. *Let $g\in AC_{loc}^{r-1}(0,\infty)$ be such that $g,\chi^r g^{(r)}\in L_p(\chi^\gamma)(0,\infty)$. Then:*

a) if $\gamma+r-1+1/p<0$ then $\lim_{x\to 0+0} g^{(j)}(x)=0$ for $0\le j<r$;

b) if $0<\gamma+i+1/p<1$ for some $i=1,2,\dots,r-1$ then $\lim_{x\to 0+0} g^{(j)}(x)=0$ for $0\le j<i$ and $\lim_{x\to\infty} g^{(j)}(x)=0$ for $i\le j<r$;

c) if $0<\gamma+1/p$ then $\lim_{x\to\infty} g^{(j)}(x)=0$ for $0\le j<r$;

d) if $\gamma=-m-1/p$ for some $m=0,1,\dots,r-1$ then $\lim_{x\to 0+0} g^{(j)}(x)=0$ for $j=0,1,\dots,m-1$ and $\lim_{x\to\infty} g^{(j)}(x)=0$ for $j=m+1,m+2,\dots,r-1$.

Note that the value $j=m$ is not considered in d).

We shall give a characterization of the weighted K-functional $K^r_{\alpha-1/p}(f,t^r)_p$ by means of K-functionals on $\mathbb{R}$ with the weight $\mathcal{E}^\alpha$. That is why, to clear that additional exponential weight, we shall need the analogue of the above inequalities for such weights.

Proposition 4.4. *(cf. [4]) Let $r\in\mathbb{N}$, $\alpha\in\mathbb{R}$ and $1\le p\le\infty$. Then for every $G\in AC_{loc}^{r-1}(\mathbb{R})$ such that $G,G^{(r)}\in L_p(\mathcal{E}^\alpha)(\mathbb{R})$ we have*

$$\|\mathcal{E}^\alpha G^{(j)}\|_{p(\mathbb{R})}\le c\left(\|\mathcal{E}^\alpha G\|_{p(\mathbb{R})}+\|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})}\right), \quad j=0,1,\dots,r,$$

where the constant c depends only on α and r.

Proof. We divide the real line by the points $a_k=k$, $k\in\mathbb{Z}$, and apply the inequality (4.1) on each interval $[a_k,a_{k+1}]$. □

Now, Proposition 4.4 and Corollary 4.2 imply

Corollary 4.5. *Let $r\in\mathbb{N}$, $\alpha\in\mathbb{R}$, $\alpha\ne 0$ and $1\le p\le\infty$. Then for every $G\in AC_{loc}^{r-1}(\mathbb{R})$ such that $G,G^{(r)}\in L_p(\mathcal{E}^\alpha)(\mathbb{R})$ we have*

$$\|\mathcal{E}^\alpha G^{(j)}\|_{p(\mathbb{R})}\le c\|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})}, \quad j=0,1,\dots,r,$$

where the constant c depends only on α and r.

Proof. It is enough to prove the statement for $j = r-1$, since the general case follows from it by iteration. Since $G, G^{(r)} \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$, then Proposition 4.4 yields that $G^{(r-1)} \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$. Now the statement follows from (4.4) with $r = 1$ and $\alpha = \gamma + 1/p \neq 0$ by the substitution $G^{(r-1)}(y) = g(e^y)$. □

5 Auxiliary relations about K-functionals

In establishing the result in Theorem 1.2, we shall first relate $K^r_\gamma(f, t^r)_p$ to the K-functional

$$\mathcal{K}^r_\alpha(F, t^r)_p = \inf_{G \in AC^{r-1}_{loc}(\mathbb{R})} \{\|\mathcal{E}^\alpha (F - G)\|_{p(\mathbb{R})} + t^r \|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})}\},$$

where $F \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$, $r \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $t > 0$. Note that the two norms in the definition of the K-functional have one and the same exponential weight.

Theorem 5.1. *Let $r \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, $0 < t \le t_0$ and $F \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$. Then*

$$\mathcal{K}^r_\alpha(F, t^r)_p \sim \omega_r(F, t)_{p(\mathcal{E}^\alpha)(\mathbb{R})}$$

where

$$\omega_r(F, t)_{p(\mathcal{E}^\alpha)(\mathbb{R})} = \sup_{0 < h \le t} \|\mathcal{E}^\alpha \Delta^r_h F\|_{p(\mathbb{R})}. \tag{5.1}$$

Proof. The proof follows the lines of its classical analogue (the case $\alpha = 0$) based upon the properties of the modulus $\omega_r(F, t)_{p(\mathcal{E}^\alpha)(\mathbb{R})}$ and the construction of modified Steklov functions (see e.g. [1, p. 177–178]). Let us note that the quantity in (5.1) is well defined since $e^{\alpha(x+h)} \sim e^{\alpha x}$ uniformly for $x \in \mathbb{R}$ and for $0 < h \le t \le t_0$, where $t_0 > 0$ is fixed. □

Definition (5.1) reduces to the classical modulus of smoothness $\omega_r(F, t)_{p(\mathbb{R})}$ in the unweighted case $\alpha = 0$.

In the proof of Theorem 6.1.b) we shall use the following characterization of a K-functional, which is a simple modification of the classical unweighted one.

Lemma 5.2. *For $r \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le t_0$ and $F \in L_p(\mathbb{R})$ there holds*

$$\inf_{G \in W^r_p(\mathbb{R})} \left\{\|F - G\|_{p(\mathbb{R})} + t^r \|G^{(r)}\|_{p(\mathbb{R})} + t^r \|G'\|_{p(\mathbb{R})}\right\} \sim \omega_r(F, t)_{p(\mathbb{R})} + t^{r-1} \omega_1(F, t)_{p(\mathbb{R})}.$$

Proof. Since for any $G \in W^r_p(\mathbb{R})$ and $0 < t \le t_0$ we have

$$\omega_r(F, t)_{p(\mathbb{R})} \le c \left(\|F - G\|_{p(\mathbb{R})} + t^r \|G^{(r)}\|_{p(\mathbb{R})}\right)$$

and

$$t^{r-1} \omega_1(F, t)_{p(\mathbb{R})} \le c \left(\|F - G\|_{p(\mathbb{R})} + t^r \|G'\|_{p(\mathbb{R})}\right)$$

there holds the lower estimate

$$\omega_r(F,t)_{p(\mathbb{R})} + t^{r-1}\omega_1(F,t)_{p(\mathbb{R})} \le c \inf_{G\in W_p^r(\mathbb{R})} \left\{ \|F-G\|_{p(\mathbb{R})} + t^r\|G^{(r)}\|_{p(\mathbb{R})} + t^r\|G'\|_{p(\mathbb{R})} \right\}.$$

To prove the converse inequality we set for any $F \in L_p(\mathbb{R})$ and $t > 0$

$$G_t(x) = \sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}\frac{1}{t^r}\int_0^t \cdots \int_0^t F\left(x + \frac{i}{r}(y_1+\cdots+y_r)\right) dy_1\cdots dy_r.$$

Then

$$\|F-G_t\|_{p(\mathbb{R})} \le \omega_r(F,t)_{p(\mathbb{R})}, \tag{5.2}$$

$$t^r\|G_t^{(r)}\|_{p(\mathbb{R})} \le c\,\omega_r(F,t)_{p(\mathbb{R})}, \tag{5.3}$$

and

$$t^r\|G_t'\|_{p(\mathbb{R})} \le c\,t^{r-1}\omega_1(F,t)_{p(\mathbb{R})}. \tag{5.4}$$

Now, inequalities (5.2) – (5.4) imply the upper estimate of the K-functional. The proof of the assertion is completed. □

From Lemma 5.2 and Proposition 4.4 with $\alpha = 0$ we get

Corollary 5.3. *For $r \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le t_0$ and $F \in L_p(\mathbb{R})$ there holds*

$$\omega_r(F,t)_{p(\mathbb{R})} + t^{r-1}\omega_1(F,t)_{p(\mathbb{R})} \le c\left(\omega_r(F,t)_{p(\mathbb{R})} + t^r\|F\|_{p(\mathbb{R})}\right).$$

6 A characterization of $K^r_{\alpha-1/p}(f,t^r)_p$ by $K-$functionals on the real line with an exponential weight

First, we establish the upper estimate.

Theorem 6.1. *Let $r \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $1 \le p \le \infty$ and $f \in L_p(\chi^{\alpha-1/p})(0,\infty)$.*

a) If $\alpha \ne 0$ and $0 < t$, then

$$K^r_{\alpha-1/p}(f,t^r)_p \le c\,\mathcal{K}^r_\alpha(f\circ\mathcal{E},t^r)_{p(\mathbb{R})}.$$

b) If $\alpha = 0$ and $0 < t \le t_0$, then

$$K^r_{-1/p}(f,t^r)_p \le c\left(\mathcal{K}^r_0(f\circ\mathcal{E},t^r)_{p(\mathbb{R})} + t^{r-1}\mathcal{K}^1_0(f\circ\mathcal{E},t)_{p(\mathbb{R})}\right).$$

Proof. For $f \in L_p(\chi^{\alpha-1/p})(0,\infty)$ set $F = f \circ \mathcal{E}$. For every $G \in AC_{loc}^{r-1}(\mathbb{R})$ such that $G, G^{(r)} \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$ we set $g = G \circ \log$. In order to prove assertion a) using the standard K-functional arguments it is enough to show that

$$\|\chi^{\alpha-1/p}(f-g)\|_{p(0,\infty)} \le c\,\|\mathcal{E}^\alpha(F-G)\|_{p(\mathbb{R})}; \tag{6.1}$$

$$\|\chi^{\alpha-1/p+r}g^{(r)}\|_{p(0,\infty)} \le c\,\|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})}. \tag{6.2}$$

Indeed, from (6.1) and (6.2) we get for every $G \in AC_{loc}^{r-1}(\mathbb{R})$ such that $G, G^{(r)} \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$ the estimate

$$\begin{aligned} K^r_{\alpha-1/p}(f,t^r)_p &\le \|\chi^{\alpha-1/p}(f-g)\|_{p(0,\infty)} + t^r\|\chi^{\alpha-1/p+r}g^{(r)}\|_{p(0,\infty)} \\ &\le c\left(\|\mathcal{E}^\alpha(F-G)\|_{p(\mathbb{R})} + t^r\|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})}\right). \end{aligned}$$

Taking infimum on G in the above inequality we get a).

By simple change of the variables we see that (6.1) is true with $c = 1$ as equality. For the proof of (6.2) we use Corollary 4.5 and get

$$\begin{aligned} \|\chi^{\alpha-1/p+r}(G\circ\log)^{(r)}\|_{p(0,\infty)} &= \left\|\chi^{\alpha-1/p+r}\chi^{-r}\sum_{j=1}^{r} m_{r,j}(G^{(j)}\circ\log)\right\|_{p(0,\infty)} \\ &\le \sum_{j=1}^{r}|m_{r,j}|\,\|\mathcal{E}^\alpha G^{(j)}\|_{p(\mathbb{R})} \le c\,\|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})} \end{aligned}$$

with appropriate integers $m_{r,j}$.

In the proof of b) we use the previous notations. Now we cannot use Corollary 4.5 in the proof of the analogue of (6.2) because $\alpha = 0$. Instead, from Proposition 4.4 with $\alpha = 0$ we get $G' \in L_p(\mathbb{R})$. Then

$$\begin{aligned} \|\chi^{-1/p+r}(G\circ\log)^{(r)}\|_{p(0,\infty)} &= \left\|\chi^{-1/p+r}\chi^{-r}\sum_{j=1}^{r} m_{r,j}(G^{(j)}\circ\log)\right\|_{p(0,\infty)} \\ &\le \sum_{j=1}^{r}|m_{r,j}|\,\|G^{(j)}\|_{p(\mathbb{R})} \\ &\le c\left(\|G'\|_{p(\mathbb{R})} + \|G^{(r)}\|_{p(\mathbb{R})}\right), \end{aligned} \tag{6.3}$$

where at the last step we use once again Proposition 4.4 with $\alpha = 0$ and G' and $r-1$ at the place of G and r. Using (6.1) with $\alpha = 0$ and (6.3) we get

$$\begin{aligned} K^r_{-1/p}(f,t^r)_p &\le c \inf_{G\in W_p^r(\mathbb{R})}\left\{\|F-G\|_{p(\mathbb{R})} + t^r\|G^{(r)}\|_{p(\mathbb{R})} + t^r\|G'\|_{p(\mathbb{R})}\right\} \\ &\le c\left(\mathcal{K}_0^r(f\circ\mathcal{E},t^r)_{p(\mathbb{R})} + t^{r-1}\mathcal{K}_0^1(f\circ\mathcal{E},t)_{p(\mathbb{R})}\right), \end{aligned}$$

where at the last step we use Lemma 5.2 and Theorem 5.1. This completes the proof. ∎

Remark 6.2. The upper estimate in the last theorem is not exact for $\alpha = 1-r, 2-r, \dots, -1$, as it follows from Remark 1.3 and Theorems 6.6.b) and 7.3 below.

Let us now proceed to the lower estimate.

Theorem 6.3. *Let $r \in \mathbb{N}$, $\alpha \neq 1-r, 2-r, \dots, -1$, $1 \le p \le \infty$, $0 < t \le t_0$ and $f \in L_p(\chi^{\alpha-1/p})(0,\infty)$. Then for $j = 1, 2, \dots, r$ there holds*

$$t^{r-j}\, \mathcal{K}_\alpha^j(f \circ \mathcal{E}, t^j)_p \le c\, K^r_{\alpha-1/p}(f, t^r)_p.$$

Proof. Let $g \in AC^{r-1}_{loc}(0,\infty)$ and $g, \chi^r g^{(r)} \in L_p(\chi^{\alpha-1/p})(0,\infty)$. We write

$$(g \circ \mathcal{E})^{(j)} = \sum_{i=1}^{j} n_{j,i} \mathcal{E}^i (g^{(i)} \circ \mathcal{E})$$

with appropriate positive integers $n_{j,i}$. Then, using Corollary 4.2 with $i = 1$ and $\gamma = \alpha - 1/p$, we get

$$\begin{aligned}
\|\mathcal{E}^\alpha (g \circ \mathcal{E})^{(j)}\|_{p(\mathbb{R})} &\le \sum_{i=1}^{j} n_{j,i}\, \|\mathcal{E}^{\alpha+i}(g^{(i)} \circ \mathcal{E})\|_{p(\mathbb{R})} \\
&= \sum_{i=1}^{j} n_{j,i}\, \|\chi^{\alpha+i-1/p} g^{(i)}\|_{p(0,\infty)} \\
&\le c\, \|\chi^{\alpha-1/p+r} g^{(r)}\|_{p(0,\infty)}.
\end{aligned}$$

Combining the above inequality with the equality $\|\mathcal{E}^\alpha (f \circ \mathcal{E} - g \circ \mathcal{E})\|_{p(\mathbb{R})} = \|\chi^{\alpha-1/p}(f-g)\|_{p(0,\infty)}$ and the condition $t \le t_0$ we complete the proof by standard K-functional arguments. ∎

Remark 6.4. In the case $r = 1$ Theorems 6.1 and 6.3 provide the equivalence

$$K^1_{\alpha-1/p}(f, t)_p \sim \mathcal{K}^1_\alpha(f \circ \mathcal{E}, t)_p$$

for all values of α.

The inequalities we have proven so far enable us to find K-functonals on the real line equivalent to $K^r_{\alpha-1/p}(f, t^r)_p$ for $\alpha \neq 1-r, 2-r, \dots, -1$. To settle the cases $\alpha = 1-r, 2-r, \dots, -1$ we shall relate them to the case $\alpha = 0$. Note that the value $\alpha = 0$ is acceptable for the hypotheses of Theorem 6.3.

Theorem 6.5. *Let $r \in \mathbb{N}$, $r \ge 2$, $m = 1, 2, \dots, r-1$, $1 \le p \le \infty$ and $f \in L_p(\chi^{-m-1/p})(0,\infty)$. Then*

$$K^r_{-m-1/p}(f, t^r)_p \sim K^r_{-1/p}(\chi^{-m} f, t^r)_p. \tag{6.4}$$

Proof. Set $F = \chi^{-m} f$. For any $G \in AC_{loc}^{r-1}(0,\infty)$ such that $G, \chi^r G^{(r)} \in L_p(\chi^{-1/p})(0,\infty)$ we set $g = \chi^m G$. From the Leibniz rule and Corollary 4.2 with $i = 1$ and $\gamma = -1/p$ we get

$$\begin{aligned}\|\chi^{-m-1/p+r} g^{(r)}\|_{p(0,\infty)} &= \|\chi^{-m-1/p+r} (\chi^m G)^{(r)}\|_{p(0,\infty)} \\ &\le \sum_{j=r-m}^{r} \binom{r}{j} \frac{m!}{(m+j-r)!} \|\chi^{-1/p+j} G^{(j)}\|_{p(0,\infty)} \\ &\le c\, \|\chi^{-1/p+r} G^{(r)}\|_{p(0,\infty)}.\end{aligned}$$

And since trivially

$$\|\chi^{-m-1/p}(f-g)\|_{p(0,\infty)} = \|\chi^{-1/p}(F-G)\|_{p(0,\infty)}, \tag{6.5}$$

we get by standard K-functional arguments

$$K^r_{-m-1/p}(f, t^r)_p \le c\, K^r_{-1/p}(F, t^r)_p.$$

The converse inequality

$$K^r_{-1/p}(F, t^r)_p \le c\, K^r_{-m-1/p}(f, t^r)_p$$

will follow from (6.5) and

$$\|\chi^{-1/p+r} G^{(r)}\|_{p(0,\infty)} \le c\, \|\chi^{-m-1/p+r} g^{(r)}\|_{p(0,\infty)}, \quad G = \chi^{-m} g, \tag{6.6}$$

valid for any $g \in AC_{loc}^{r-1}(0,\infty)$ such that $g, \chi^r g^{(r)} \in L_p(\chi^{-m-1/p})(0,\infty)$. By the Leibniz rule we have

$$G^{(r)}(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{(m+r-j-1)!}{(m-1)!}\, x^{-m-r+j} g^{(j)}(x). \tag{6.7}$$

If $m < r-1$ then we observe that Corollary 4.2 with $i = m+1$ and $\gamma = -m-1/p$ implies for $j = m+1, \dots, r-1$

$$\|\chi^{-m-1/p+j} g^{(j)}\|_{p(0,\infty)} \le c\, \|\chi^{-m-1/p+r} g^{(r)}\|_{p(0,\infty)}. \tag{6.8}$$

We shall show that

$$\begin{aligned}\Big\|\chi^{-m-1/p} \sum_{j=0}^{m} (-1)^{r-j} \binom{r}{j} (m+r-j-1)!\, \chi^j g^{(j)}\Big\|_{p(0,\infty)} \\ \le c\, \|\chi^{1-1/p} g^{(m+1)}\|_{p(0,\infty)}.\end{aligned} \tag{6.9}$$

Then (6.7) – (6.9) imply (6.6) as (6.8) is not necessary in case $m = r-1$. So it remains to prove (6.9).

First, putting $g(x) = x^m$ in (6.7), we get

$$0 \equiv \sum_{j=0}^{m} (-1)^{r-j} \binom{r}{j} \frac{(m+r-j-1)!}{(m-1)!}\, x^{-m-r+j} \frac{m!}{(m-j)!}\, x^{m-j}$$

and hence

$$\sum_{j=0}^{m}(-1)^{r-j}\binom{r}{j}\frac{(m+r-j-1)!}{(m-j)!}=0. \tag{6.10}$$

Next, we expand $g^{(j)}$, $j=0,1,\dots,m$, by the Taylor expansion at the point $u>0$ up to the derivative of order $m+1$ and after rearranging the summands according to the order of the derivatives, we get

$$\begin{aligned}
&\sum_{j=0}^{m}(-1)^{r-j}\binom{r}{j}(m+r-j-1)!\,x^j g^{(j)}(x)\\
&\quad=\sum_{\ell=0}^{m}\left[\sum_{j=0}^{\ell}(-1)^{r-j}\binom{r}{j}\frac{(m+r-j-1)!}{(\ell-j)!}\,x^j\,(x-u)^{\ell-j}\right]g^{(\ell)}(u)\\
&\qquad+\sum_{j=0}^{m}(-1)^{r-j}\binom{r}{j}\frac{(m+r-j-1)!}{(m-j)!}\,x^j\int_u^x(x-y)^{m-j}g^{(m+1)}(y)\,dy.
\end{aligned}$$

Now, taking into consideration (6.10), we get

$$\begin{aligned}
&\sum_{j=0}^{m}(-1)^{r-j}\binom{r}{j}(m+r-j-1)!\,x^j g^{(j)}(x)\\
&\quad=\sum_{\ell=0}^{m-1}\left[\sum_{j=0}^{\ell}(-1)^{r-j}\binom{r}{j}\frac{(m+r-j-1)!}{(\ell-j)!}\,x^j\,(x-u)^{\ell-j}\right]g^{(\ell)}(u)\\
&\qquad+\left[\sum_{k=1}^{m}(-1)^k\mu_{r,m,k}x^{m-k}u^{k-1}\right]u\,g^{(m)}(u)\\
&\qquad+\sum_{k=1}^{m}(-1)^k\mu_{r,m,k}\,x^{m-k}\int_u^x y^k g^{(m+1)}(y)\,dy,
\end{aligned} \tag{6.11}$$

where for $k=1,2,\dots,m$ we have put

$$\begin{aligned}
\mu_{r,m,k}&=\sum_{j=0}^{m-k}(-1)^{r-j}\binom{r}{j}\frac{(m+r-j-1)!}{(m-j)!}\binom{m-j}{k}\\
&=(-1)^r\binom{m-1}{m-k}\frac{(r+k-1)!}{k!}.
\end{aligned}$$

In order to get a simpler representation than (6.11), we shall take the limit $u\to 0+0$. Before that we emphasize on three facts. It was established in Corollary 4.2 d) that

$$\lim_{u\to 0+0} g^{(\ell)}(u)=0,\quad \ell=0,1,\dots,m-1. \tag{6.12}$$

Since

$$u\,g^{(m)}(u)=u\,g^{(m)}(1)+u\int_1^u g^{(m+1)}(y)\,dy$$

and Hölder's inequality gives

$$\left| u \int_1^u g^{(m+1)}(y)\,dy \right| \le u|\log u|^{1-1/p} \|\chi^{1-1/p} g^{(m+1)}\|_{p(0,\infty)}$$

we get

$$\lim_{u \to 0+0} u\, g^{(m)}(u) = 0. \tag{6.13}$$

From $\chi^{1-1/p} g^{(m+1)} \in L_p[0,1]$ and $\chi^{1/p} \in L_\infty[0,1]$ we get

$$\chi\, g^{(m+1)} \in L_1[0,1]. \tag{6.14}$$

Now, taking the limit $u \to 0+0$ in (6.11) (for an arbitrary fixed positive x) and having in mind (6.12) – (6.14), we get the representation

$$\sum_{j=0}^{m} (-1)^{r-j} \binom{r}{j} (m+r-j-1)!\, x^j g^{(j)}(x) = \sum_{k=1}^{m} (-1)^k \mu_{r,m,k}\, x^{m-k} \int_0^x y^k g^{(m+1)}(y)\,dy.$$

Finally, Hardy's inequality applied to the right-hand side of the above formula implies (6.9). This completes the proof of the theorem. □

Combining the results from Theorems 6.1, 6.3, 6.5 and 5.1 we get

Theorem 6.6. *Let $r \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, $0 < t \le t_0$ and $f \in L_p(\chi^{\alpha-1/p})(0,\infty)$.*

a) If $\alpha \ne 1-r, 2-r, \ldots, -1, 0$, then

$$K^r_{\alpha-1/p}(f, t^r)_p \sim \mathcal{K}^r_\alpha(f \circ \mathcal{E}, t^r)_p \sim \omega_r(f \circ \mathcal{E}, t)_{p(\mathcal{E}^\alpha)(\mathbb{R})}.$$

b) If $\alpha = 1-r, 2-r, \ldots, -1, 0$, then

$$K^r_{\alpha-1/p}(f, t^r)_p \sim \omega_r((\chi^\alpha f) \circ \mathcal{E}, t)_{p(\mathbb{R})} + t^{r-1} \omega_1((\chi^\alpha f) \circ \mathcal{E}, t)_{p(\mathbb{R})}.$$

Remark 6.7. The second term in the relation in b) cannot be dropped or replaced by a modulus of different order of the same function as it was shown in Remark 1.3.

Remark 6.8. Although

$$K^r_{\alpha-1/p}(f, t^r)_p \le c\, \omega_r(f \circ \mathcal{E}, t)_{p(\mathcal{E}^\alpha)(\mathbb{R})}$$

in the cases $\alpha = 1-r, 2-r, \ldots, -1$ as well, the converse inequality is not valid for these values of α. For the sake of simplicity we shall consider only the case $p = \infty$. Let $\alpha = -m$, where $m \in \{1, 2, \ldots, r-1\}$. Then for $f_m(x) = x^m$ we have $f_m \in C(\chi^{-m})(0,\infty)$, $K^r_{-m-1/p}(f_m, t^r)_\infty \equiv 0$ while $\omega_r(f_m \circ \mathcal{E}, t)_{\infty(\mathcal{E}^{-m})(\mathbb{R})} = (e^{tm} - 1)^r \ne 0$.

7 A characterization of $\mathcal{K}_\alpha^r(F,t^r)_p$ by the classical moduli of smoothness

Again first we shall establish the upper estimate.

Theorem 7.1. *Let $r \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $1 \le p \le \infty$. Then for $F \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$ and $0 < t \le t_0$ we have*

$$\mathcal{K}_\alpha^r(F,t^r)_p \le c\left(\mathcal{K}_0^r(\mathcal{E}^\alpha F,t)_p + t^r\,\|\mathcal{E}^\alpha F\|_{p(\mathbb{R})}\right).$$

Proof. Let $g \in W_p^r(\mathbb{R})$ be arbitrary. Then the Leibniz rule gives

$$(e^{-\alpha x}g(x))^{(r)} = \sum_{i=0}^{r}\binom{r}{i}(-\alpha)^{r-i}e^{-\alpha x}g^{(i)}(x) \tag{7.1}$$

and hence for $G = \mathcal{E}^{-\alpha}g$ using Proposition 4.4 with $\alpha = 0$ we get

$$\begin{aligned}\|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})} &\le \sum_{i=0}^{r}\binom{r}{i}|\alpha|^{r-i}\|g^{(i)}\|_{p(\mathbb{R})}\\ &\le c\left(\|g\|_{p(\mathbb{R})} + \|g^{(r)}\|_{p(\mathbb{R})}\right)\\ &\le c\left(\|\mathcal{E}^\alpha F - g\|_{(\mathbb{R})} + \|g^{(r)}\|_{p(\mathbb{R})} + \|\mathcal{E}^\alpha F\|_{p(\mathbb{R})}\right).\end{aligned}$$

Since also $\|\mathcal{E}^\alpha(F-G)\|_{p(\mathbb{R})} = \|\mathcal{E}^\alpha F - g\|_{p(\mathbb{R})}$ the standard K-functional arguments prove the theorem. □

The lower estimate is given in the next theorem.

Theorem 7.2. *Let $r \in \mathbb{N}$, $\alpha \neq 0$ and $1 \le p \le \infty$. Then for $F \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$, $0 < t \le t_0$ and $j = 0, 1, \ldots, r$ there holds*

$$t^{r-j}\,\mathcal{K}_0^j(\mathcal{E}^\alpha F,t^j)_p \le c\,\mathcal{K}_\alpha^r(F,t^r)_p,$$

where we have set $\mathcal{K}_0^0(f,1)_{p(\mathbb{R})} = \|f\|_{p(\mathbb{R})}$.

Proof. Let $G \in AC_{loc}^{r-1}(\mathbb{R})$ such that $G, G^{(r)} \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$ be arbitrary. From (7.1) with α and j instead of $-\alpha$ and r and Corollary 4.5 we get

$$\|(\mathcal{E}^\alpha G)^{(j)}\|_{p(\mathbb{R})} \le \sum_{i=0}^{j}\binom{j}{i}|\alpha|^{j-i}\|\mathcal{E}^\alpha G^{(i)}\|_{p(\mathbb{R})} \le c\,\|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})}.$$

Hence

$$\begin{aligned}t^{r-j}\,\mathcal{K}_0^j(\mathcal{E}^\alpha F,t^j)_p &\le t^{r-j}\,\|\mathcal{E}^\alpha F - \mathcal{E}^\alpha G\|_{p(\mathbb{R})} + t^r\|(\mathcal{E}^\alpha G)^{(j)}\|_{p(\mathbb{R})}\\ &\le c\left(\|\mathcal{E}^\alpha(F-G)\|_{p(\mathbb{R})} + t^r\|\mathcal{E}^\alpha G^{(r)}\|_{p(\mathbb{R})}\right),\end{aligned}$$

which proves the theorem by taking infimum on G. □

Now, Theorems 7.1, 7.2 and 5.1 with $\alpha = 0$ give the characterization

Theorem 7.3. *Let $r \in \mathbb{N}$, $\alpha \neq 0$ and $1 \leq p \leq \infty$. Then for $F \in L_p(\mathcal{E}^\alpha)(\mathbb{R})$ and $0 < t \leq t_0$ we have*

$$\mathcal{K}_\alpha^r(F, t^r)_p \sim \omega_r(\mathcal{E}^\alpha F, t)_{p(\mathbb{R})} + t^r \, \|\mathcal{E}^\alpha F\|_{p(\mathbb{R})}.$$

Remark 7.4. The additional term in the characterization above cannot be dropped or replaced by a modulus of smoothness of the function $\mathcal{E}^\alpha F$ as we observed in Remark 1.3.

The last theorem implies the following relation between K-functionals of the class $\mathcal{K}_\alpha^r(F, t^r)_p$, $\alpha \neq 0$.

Corollary 7.5. *Let $r \in \mathbb{N}$, $\alpha_1, \alpha_2 \neq 0$ and $1 \leq p \leq \infty$. Then for $F \in L_p(\mathcal{E}^{\alpha_1})(\mathbb{R})$ and $0 < t \leq t_0$ we have*

$$\mathcal{K}_{\alpha_1}^r(F, t^r)_p \sim \mathcal{K}_{\alpha_2}^r(\mathcal{E}^{\alpha_1 - \alpha_2} F, t^r)_p.$$

Remark 7.6. Consider the space

$$C(\chi^\gamma)[0, \infty) = \{f \,:\, \chi^\gamma f \in C(0, \infty), \ \exists \lim_{x \to 0+0} \chi^\gamma f\}.$$

For functions $f \in C(\chi^\gamma)[0, \infty)$ we may define a slightly different functional than (1.5) imposing the additional restriction $g \in C(\chi^\gamma)[0, \infty)$ on the functions g on which the infimum is taken. Let us denote this K-functional by

$$K(f, t^r; C(\chi^\gamma)[0, \infty), AC_{loc}^{r-1}, \varphi^r D^r).$$

Theorem 1.2 with $p = \infty$ holds for this K-functional too. This fact follows from the equivalence

$$\begin{aligned} K(f, t^r; C(\chi^\gamma)(0, \infty), AC_{loc}^{r-1}, \varphi^r D^r) &\leq K(f, t^r; C(\chi^\gamma)[0, \infty), AC_{loc}^{r-1}, \varphi^r D^r) \\ &\leq c\, K(f, t^r; C(\chi^\gamma)(0, \infty), AC_{loc}^{r-1}, \varphi^r D^r), \end{aligned}$$

valid for $r \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $f \in C(\chi^\gamma)[0, \infty)$. The first inequality is obvious – an infimum on a more narrow class is taken in the second K-functional. The second inequality follows by a careful examination of the proofs of Theorems 2.1.1 and 6.1.1 in [3] – the functions G_t there belong to $C(\chi^\gamma)[0, \infty)$ if f does.

The same observations are true if we require $\chi^\gamma f$ to have limit at ∞ or to have limits at 0 and at ∞.

References

[1] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Berlin, Springer-Verlag, 1993.

[2] Z. Ditzian, K.G. Ivanov, Strong converse inequalities, J. Anal. Math. 61 (1993) 61-111.

[3] Z. Ditzian, V. Totik, Moduli of Smoothness, New York, Springer Verlag, 1987.

[4] Z. Ditzian, V. Totik, K-functionals and weighted moduli of smoothness, J. Approx. Theory 63 (1990) 3-29.

[5] B.R. Draganov, K.G. Ivanov, A new characterization of weighted Peetre K-functionals, Constr. Approx. 21 (2005) 113-148.

[6] B.R. Draganov, K.G. Ivanov, A characterization of weighted L_p approximations by the Gamma and the Post-Widder operators, in: Proc. intern. conf. "Constructive Theory of Functions, Varna 2005" (to appear).

[7] S. Guo, L. Liu, Q. Qi, G. Zhang, A strong converse inequality for left gamma quasi-interpolants in L_p-spaces, Acta Math. Hungar. 105, 1-2 (2004) 17-26.

[8] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, 2nd ed., Cambridge Univ. Press, 1951.

[9] A. Lupas, M. Müller, Approximationseigenschaften der Gammaoperatoren, Math. Z. 98 (1967) 208-226.

[10] Q.-L. Qi, S.-S. Guo, Strong converse inequality for left gamma quasi-interpolants, Acta Math. Appl. Sinica 21, 1 (2005) 115-124.

[11] C. Sangüesa, Lower estimates for centered Bernstein-type operators, Constr. Approx. 18 (2002) 145-159.

[12] V. Totik, The gammaoperators in L^p spaces, Publ. Math. (Debrecen) 32 (1985) 43-55.

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