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EQUIVALENCE BETWEEN *K*-FUNCTIONALS BASED ON CONTINUOUS LINEAR TRANSFORMS

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Abstract. The paper presents a method of relating two K-functionals by means of a continuous linear transform of the function. In particular, a characterization of various weighted K-functionals by unweighted fixed-step moduli of smoothness is derived. This is applied in estimating the rate of convergence of several approximation processes.

1. *K*-functionals in measuring the approximation error.

Finding a good estimate of the error of a given approximation process is a basic problem in approximation theory. The so called K-functional turned out to be

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very useful in this respect. It has been introduced by Lions and by Peetre and in its present form by Peetre [28], as a basis of the theory of interpolation of operators (e.g. [2, Ch. 5]). Later Butzer and Berens [4] clarified its importance in approximation theory.

Let X be a Banach space with norm $\|\cdot\|_X$ and Y be another space with a semi-norm $|\cdot|_Y$. For $f \in X$ and $t > 0$ we define the K-functional between the spaces X and Y by

$$K(f, t; X, Y) = \inf\{\|f - g\|_X + t|g|_Y : g \in Y \cap X\}. \tag{1.1}$$

Usually Y is a dense subspace of X and consists of elements that possess certain additional properties as, for example, high smoothness. As it is seen from its definition the K-functional measures how well a given function $f \in X$ can be approximated by elements $g \in Y$ with control of their semi-norm $|g|_Y$. For given $f \in X$ the K-functional is a non-negative and non-decreasing function of t. Thus, $\lim_{t\to 0+} K(f, t; X, Y)$ always exists. Moreover, this limit is 0 for every $f \in X$ iff Y is dense in X.

For many applications in approximation theory X is a weighted L_p-space $L_p(w)(I) = \{f \in L_{1,loc}(I) : wf \in L_p(I)\}$, $1 \le p \le \infty$, with a norm $\|f\|_X = \|wf\|_{p(I)}$ and Y is a weighted Sobolev space $W_p^r(\phi)(I) = \{g \in AC_{loc}^{r-1}(I) : \phi g^{(r)} \in L_p(I)\}$ with a semi-norm $|g|_Y = \|\phi g^{(r)}\|_{p(I)}$, where I is an interval on the real line and w, ϕ are measurable on it with singularities only at its ends.

A standard application of the K-functional is the direct theorem for the error of an approximation process. If $\{\mathcal{L}_\alpha\}_{\alpha\in A}$ is a family of linear operators that maps the Banach space X into itself such that

a) $\|\mathcal{L}_\alpha f\|_X \le c\,\|f\|_X$ for every $f \in X$ and $\alpha \in A$,

b) $\|g - \mathcal{L}_\alpha g\|_X \le c\,\theta(\alpha)\,|g|_Y$ for every $g \in Y$ and $\alpha \in A$ with $\theta : A \to \mathcal{R}_+$,

then for every $f \in X$ and $\alpha \in A$ there holds the estimate

$$\|f - \mathcal{L}_\alpha f\|_X \le c\,K(f, \theta(\alpha); X, Y). \tag{1.2}$$

Above and in what follows we denote by c positive constants not necessarily the same on each occurrence that do not depend on the function f and the parameter t in the K-functional.

Usually the estimates converse to (1.2) are of weak type but in a number of cases connected with saturated approximation processes strong type inverse inequalities can be established. Let $\mathcal{D}$ be a differential operator, $Y = \{g \in X :$

$\mathcal{D}g \in X\}$ and $|g|_Y = \|\mathcal{D}g\|_X$. Ditzian and Ivanov proved in [9] that the inverse inequality to (1.2)

$$K(f,\theta(\alpha);X,Y) \le c\,\|f-\mathcal{L}_\alpha f\|_X \tag{1.3}$$

follows from a) and the inequalities:

c) $\|g-\mathcal{L}_\alpha g-\theta(\alpha)\mathcal{D}g\|_X \le \psi(\alpha)\,|g|_Z$ for every $g\in Z$,

d) $\|\mathcal{D}\mathcal{L}_\alpha^k f\|_X \le \dfrac{c}{\theta(\alpha)}\,\|f\|_X$ for every $f\in X$,

e) $|\mathcal{L}_\alpha^\ell g|_Z \le M\,\dfrac{\theta(\alpha)}{\psi(\alpha)}\,\|\mathcal{D}\mathcal{L}_\alpha^k g\|_X$ for every $g\in X$,

where $Z\subset X$, $\theta : A\to\mathcal{R}_+$ is bounded, $\psi : A\to\mathcal{R}_+$, $k,\ell\in\mathcal{N}$, $k\le\ell$, and $M<1$ is a constant.

Thus, inequalities (1.2) and (1.3) imply that the approximation error and the K-functional are equivalent, which we denote by

$$\|f-\mathcal{L}_\alpha f\|_X \sim K(f,\theta(\alpha);X,Y). \tag{1.4}$$

So, using K-functionals we can derive an estimate of the approximation error from several inequalities. However, let us note that in general it is difficult to establish some of inequalities c)-e), especially e) with a *small* constant on the right-hand side. The error estimates through K-functionals are of high importance in approximation theory but they solely are of little effectiveness because it is difficult to evaluate for a given f and every $t\in(0,t_0]$ the infimum *on the wide space* Y. This shortcoming can be overcome by defining a new functional characteristic $\Omega(f,t)$, called modulus of smoothness, which is equivalent to the K-functional. The modulus of smoothness depends on f more directly and can be more easily estimated being a supremum or an average on a neighbourhood of the origin *in a finite dimensional space*.

2. Moduli of smoothness. Let $X = L_p(I)$, $I\subseteq\mathcal{R}$ is an interval, $1\le p\le\infty$, with the usual L_p-norm on the interval I, and $Y = W_p^r(I) = \{g\in AC_{loc}^{r-1}(I) : g^{(r)}\in L_p(I)\}$ – the Sobolev space with the semi-norm $|g|_{W_p^r(I)} = \|g^{(r)}\|_{p(I)}$. It is well known (see [24, 25]) that for every $f\in L_p(I)$ and $0<t\le t_0$ we have

$$K(f,t^r;L_p(I),W_p^r(I)) \sim \omega_r(f,t)_{p(I)}, \tag{2.1}$$

where $\omega_r(f,t)_{p(I)}$ is the classical unweighted fixed-step modulus of smoothness of order r of the function f, namely,

$$\omega_r(f,t)_{p(I)} = \sup_{0<h\le t} \|\Delta_h^r f\|_{p(I)}$$

and $\Delta_h^r f(x)$ is the finite difference of the function f of order r and step h. We assume that $\Delta_h^r f(x) = 0$ if the argument of any of the summands of the finite differences $\Delta_h^r f(x)$ is outside I. Thus, if we consider, for example, symmetric finite differences:

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r/2 - k)h) \tag{2.2}$$

of functions $f \in L_p[a,b]$, where $[a,b]$ is a finite interval, we have

$$\omega_r(f,t)_{p[a,b]} = \sup_{0<h\le t} \|\Delta_h^r f\|_{p[a+rh/2,b-rh/2]}.$$

We also set $\omega_0(f,t)_{p(I)} = \|f\|_{p(I)}$. Let us note that for 2π-periodic functions the r-th modulus $\omega_r(f,t)_{p,2\pi}$ is defined in a slightly different way – the norm is taken on an arbitrary period but the convention for vanishing of the finite difference is not applied.

Let w and φ be power-type weights with singularities only at the ends of the interval $I \subseteq \mathcal{R}$. Ditzian and Totik [10, pp. 56, 218] introduced the varying-step moduli of smoothness, which for a finite interval $I = [a,b]$ are defined by

$$\begin{aligned} \omega_\varphi^r(f,t)_{w,p[a,b]} = {} & \sup_{0<h\le t} \|w\Delta_{h\varphi}^r f\|_{p[a+t_a^*,b-t_b^*]} \\ & + \sup_{0<h\le t_a^*} \|w\overrightarrow{\Delta}_h^r f\|_{p[a,a+12t_a^*]} + \sup_{0<h\le t_b^*} \|w\overleftarrow{\Delta}_h^r f\|_{p[b-12t_b^*,b]}, \end{aligned} \tag{2.3}$$

where $\overrightarrow{\Delta}_h^r$ and $\overleftarrow{\Delta}_h^r$ denote forward and backward r-th finite differences respectively and t_a^*, t_b^* are functions of t and r depending on the behavior of φ at the respective end-points.

Under certain conditions, the most important of which are the "finite overlapping condition" on φ and the boundness of w at a finite end-point of I, Ditzian and Totik proved in [10, Ch. 2 and Ch. 6] the equivalence

$$K(f,t^r; L_p(w)(I), W_p^r(w\varphi^r)(I)) \sim \omega_\varphi^r(f,t)_{w,p(I)},$$

where the K-functional is defined for $X = L_p(w)(I)$, $1 \le p \le \infty$, and $Y = W_p^r(w\varphi^r)(I)$ with the semi-norm $|g|_{W_p^r(w\varphi^r)(I)} = \|w\varphi^r g^{(r)}\|_{p(I)}$.

On the other hand, the second author introduced the following moduli of smoothness

$$\tau_r(f;\psi(t))_{q,p(I)} = \|\omega_r(f,\cdot;\psi(t,\cdot))_q\|_{p(I)}, \tag{2.4}$$

where the local moduli are given by

$$\omega_r(f,x;\psi(t,x))_q = \left((2\psi(t,x))^{-1}\int_{-\psi(t,x)}^{\psi(t,x)} |\Delta_h^r f(x)|^q\,dh\right)^{1/q},\ 1\le q<\infty,$$

$$\omega_r(f,x;\psi(t,x))_\infty = \sup\{|\Delta_h^r f(x)| : |h|\le\psi(t,x)\}$$

and ψ is a continuous function connected with φ in a certain way (see [21, 22]). Under certain conditions on φ (see [22]), Ivanov proved the relation

$$K(f,t^r;L_p(I),W_p^r(\varphi^r)(I)) \sim \tau_r(f;\psi(t))_{p,p(I)}.$$

The power and logarithmic-type weights φ are covered. The first characterization, based on the standard translation operator, of the best approximations in $L_p[a,b]$, $1\le p\le\infty$, by algebraic polynomials was established in the terms of τ-moduli [21].

Let us also mention that Ky [27] defined moduli through which he characterized K-functionals of the type (1.1) with $X = L_p(w)(I)$, $I\subseteq\mathcal{R}$ is an interval, $1\le p\le\infty$, and $Y = W_p^r(w)(I)$, where the weight w is bounded and satisfies certain monotonicity requirements near the end-points of the interval.

M. K. Potapov (see [29] and the references cited there) characterized the K-functional (1.1) with $X = L_p(w)[-1,1]$ with Jacobean weight w, and $Y = \{g\in AC_{loc}^{2r-1}[-1,1] : D_{\nu,\mu}^r g\in L_p[-1,1]\}$ with the semi-norm $|g|_Y = \|D_{\nu,\mu}^r g\|_{p[-1,1]}$, where

$$D_{\nu,\mu} = (1-x)^{-\nu}(1+x)^{-\mu}\frac{d}{dx}(1-x)^{\nu+1}(1+x)^{\mu+1}\frac{d}{dx}$$

is the Jacobean differential operator. The moduli introduced by Potapov are based on generalized translation operators.

Also Butzer, Stens and Wehrens [3, 5, 6] defined moduli of smoothness by means of multipliers and generalized translations to characterize the best weighted algebraic approximation with Jacobean weights.

3. Equivalence between *K*-functionals. The main purpose of this paper is to present a general approach to establishing equivalence between two K-functionals of the type:

$$K(f,t;X_1,Y_1) \sim K(\mathcal{A}f,t;X_2,Y_2),$$

where $\mathcal{A} : X_1 \to X_2$ is a bounded linear operator. Hence, in view of (2.1) for $X_2 = L_p(I)$, $Y_2 = W_p^r(I)$, $|G|_{W_p^r(I)} = \|G^{(r)}\|_{p(I)}$, $I \subseteq \mathcal{R}$ is an interval, we get the following characterization of the K-functional $K(f,t;X_1,Y_1)$ by means of the unweighted fixed step-modulus of smoothness:

$$K(f,t^r;X_1,Y_1) \sim \omega_r(\mathcal{A}f,t)_{p(I)}.$$

We have (cf. [16, Definition 2.1 and Proposition 2.1] and [13, Definition 1.1 and Proposition 2.1])

Theorem 3.1. *If there exists a linear operator* $\mathcal{B} : X_2 \to X_1$*, related to* $\mathcal{A} : X_1 \to X_2$*, and both operators satisfy the conditions:*

a) $\|\mathcal{A}f\|_{X_2} \le c\,\|f\|_{X_1}$ *for every* $f \in X_1$*;*

b) $\mathcal{A}g \in Y_2 \cap X_2$ *and* $|\mathcal{A}g|_{Y_2} \le c\,|g|_{Y_1}$ *for every* $g \in Y_1 \cap X_1$*;*

c) $\|\mathcal{B}F\|_{X_1} \le c\,\|F\|_{X_2}$ *for every* $F \in X_2$*;*

d) $\mathcal{B}G \in Y_1 \cap X_1$ *and* $|\mathcal{B}G|_{Y_1} \le c\,|G|_{Y_2}$ *for any* $G \in Y_2 \cap X_2$*;*

e) $|f - \mathcal{B}\mathcal{A}f|_{Y_1} = 0$ *for every* $f \in X_1$*;*

f) $|F - \mathcal{A}\mathcal{B}F|_{Y_2} = 0$ *for every* $F \in X_2$*.*

Then

$$K(f,t;X_1,Y_1) \sim K(\mathcal{A}f,t;X_2,Y_2) \tag{3.1}$$

and

$$K(F,t;X_2,Y_2) \sim K(\mathcal{B}F,t;X_1,Y_1).$$

Remark 3.1. Let us note that to get only (3.1) it is sufficient condition f) to be fulfilled only for $F \in \mathcal{A}(X_1)$.

In some cases in order to apply Theorem 3.1 more effectively in establishing a characterization of K-functionals $K(f,t^r;L_p(w)(I),W_p^r(w\varphi^r)(I))$ by the unweighted fixed-step moduli we shall split the singularities of the weights. More precisely, let $-\infty \le \bar{a} < a_1 < b_1 < \bar{b} \le \infty$, $I = (\bar{a},\bar{b})$, $I_1 = (\bar{a},b_1)$ and $I_2 = (a_1,\bar{b})$. Let w and φ be non-negative measurable on I weights such that $w \sim 1$ and $\varphi \sim 1$

on $[a_1, b_1]$. Then for $r \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le b_1 - a_1$ and $f \in L_p(w)(I)$ we have (cf. [8, p. 176, Lemma 2.3] and [16, Lemma 7.1]):

$$\begin{aligned} K(f,t;L_p(w)(I),W_p^r(w\varphi^r)(I)) &\sim K(f,t;L_p(w)(I_1),W_p^r(w\varphi^r)(I_1)) \\ &\quad + K(f,t;L_p(w)(I_2),W_p^r(w\varphi^r)(I_2)). \end{aligned} \tag{3.2}$$

Above we have used one and the same notation for the function f and for its restrictions on subdomains.

Let us note that in the applications we require operator $\mathcal{A}$ from Theorem 3.1 to be constructed explicitly and the computations of $\omega_r(\mathcal{A}f,t)_p$ and $\omega_r(f,t)_p$ to be equally difficult. The latter means that the operations in constructing $\mathcal{A}$ are addition and multiplication by elementary functions, change of the variable by elementary functions and integration.

4. Application. Now, we give several applications of Theorem 3.1 in establishing characterizations of K-functionals in terms of the unweighted fixed-step moduli of smoothness.

Let us consider the K-functional

$$\begin{aligned} &K(f,t;L_p(w)(I),W_p^r(w\varphi^r)(I)) \\ &\qquad = \inf\left\{ \|w(f-g)\|_{p(I)} + t\|w\varphi^r g^{(r)}\|_{p(I)} \ : \ g \in AC_{loc}^{r-1}(I) \right\}. \end{aligned} \tag{4.1}$$

The domain I may be finite, semi-infinite or infinite, represented respectively by $I = [a,b]$, $I = [a,\infty)$ and $I = (-\infty,\infty)$. To define the weights w and φ we set $\chi_\xi(x) = |x-\xi|$ for a real number ξ. For a finite domain $I = [a,b]$ we consider the weights $w = \chi_a^{\gamma_a}\chi_b^{\gamma_b}$ and $\varphi = \chi_a^{\lambda_a}\chi_b^{\lambda_b}$ with $\gamma_a, \gamma_b, \lambda_a, \lambda_b \in \mathbb{R}$. For a semi-infinite domain $I = [a,\infty)$ we consider the weights $w = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ (note that $w(x)/x^{\gamma_\infty} \to 1$ for $x \to \infty$) and $\varphi = \chi_a^{\lambda_a}\chi_{a-1}^{\lambda_\infty-\lambda_a}$ with $\gamma_a, \gamma_\infty, \lambda_a, \lambda_\infty \in \mathbb{R}$. For the infinite domain $I = (-\infty,\infty)$ we apply (3.2) to reduce the case to semi-infinite domain.

It is demonstrated in [16] that the case $\lambda_a > 1$ with a finite end-point a of the domain I is equivalent to the case $\lambda_\infty < 1$ (transfer to infinite end-point), as well as the case $\lambda_\infty > 1$ is equivalent to the case $\lambda_a < 1$. So the two main cases in characterization of the K-functional (4.1) are i) $\lambda_a < 1$ and $\lambda_\infty < 1$ (see Subsection 4.1) and ii) $\lambda_a = 1$ and $\lambda_\infty = 1$ (see Subsection 4.2). In order to solve the first case we apply power change of the variable and for the second case – exponential change of the variable. In the remaining subsections we consider applications in which the differential operator $\varphi^r D^r$ determining the second term of the K-functional is replaced by a linear differential operator of the form $P(D)$, where P is a polynomial with constant coefficients in Subsection 4.3 or with varying coefficients in Subsection 4.4.

Also to describe the restrictions on the powers γ_a, γ_b or γ_∞ of the weight w we set for $r \in \mathcal{N}$ and $1 \le p \le \infty$

$$\begin{aligned}
&\Gamma_+(p) = (-1/p, \infty),\ p < \infty,\ \text{and } \Gamma_+(\infty) = [0, \infty);\\
&\Gamma_0(p) = (-1/p, \infty);\\
&\Gamma_i(p) = (-i - 1/p, 1 - i - 1/p), \quad i = 1, \dots, r-1;\\
&\Gamma_r(p) = (-\infty, 1 - r - 1/p);\\
&\Gamma_{exc}(p) = \{1 - r - 1/p, 2 - r - 1/p, \dots, -1/p\}.
\end{aligned}$$

4.1. Power change of the variable. K-functionals with $\lambda_a < 1$ and/or $\lambda_\infty < 1$ are related to the best approximation by algebraic polynomials on a finite interval, to the approximation error of the Bernstein, Szasz-Mirakian and Baskakov operators, etc. [10, 21, 26, 31].

Let $r \in \mathcal{N}$ and $\xi \in (a, b)$. Let s be one of the ends of the finite interval $[a, b]$ and e – the other.

For $\rho \in \mathcal{R}$, $i \in \mathcal{N}_0$, $i \le r$, $x \in (a, b)$ and $f \in L_{1,loc}[a, b]$, satisfying the additional requirement $\chi_s^{-i+\rho} f \in L_1[s, (s+e)/2]$ if $i > 0$, we set

$$\begin{aligned}
(A_i(\rho; s, e; \xi) f)(x) = {}& \left(\frac{x-s}{e-s}\right)^{\rho} f(x)\\
&+ \frac{1}{e-s} \sum_{k=1}^{i} \alpha_{r,k}(\rho) \left(\frac{x-s}{e-s}\right)^{k-1} \int_s^x \left(\frac{y-s}{e-s}\right)^{-k+\rho} f(y)\, dy\\
&+ \frac{1}{e-s} \sum_{k=i+1}^{r} \alpha_{r,k}(\rho) \left(\frac{x-s}{e-s}\right)^{k-1} \int_\xi^x \left(\frac{y-s}{e-s}\right)^{-k+\rho} f(y)\, dy,
\end{aligned}$$

where

$$\alpha_{r,k}(\rho) = \frac{(-1)^k}{(r-1)!} \binom{r-1}{k-1} \prod_{\nu=0}^{r-1} (\rho + r - k - \nu), \quad k = 1, 2, \dots, r.$$

As usual, above and in what follows we assume that the sum is 0 if the upper bound is smaller than the lower.

For $\sigma > 0$, $i \in \mathbb{N}$, $i \leq r$, $x \in (a, b)$ and $f \in L_{1,loc}[a, b]$, satisfying the additional requirement $\chi_s^{(1-i)/\sigma-1} f \in L_1[s, (s+e)/2]$ if $i > 1$, we set

$$(B_i(\sigma; s, e; \xi)f)(x) = f\left(s + (e-s)\left(\frac{x-s}{e-s}\right)^{\sigma}\right)$$
$$+ \frac{1}{e-s} \sum_{k=2}^{i} \beta_{r,k}(\sigma) \left(\frac{x-s}{e-s}\right)^{k-1} \int_s^x \left(\frac{y-s}{e-s}\right)^{-k} f\left(s + (e-s)\left(\frac{y-s}{e-s}\right)^{\sigma}\right) dy,$$
$$+ \frac{1}{e-s} \sum_{k=i+1}^{r} \beta_{r,k}(\sigma) \left(\frac{x-s}{e-s}\right)^{k-1} \int_\xi^x \left(\frac{y-s}{e-s}\right)^{-k} f\left(s + (e-s)\left(\frac{y-s}{e-s}\right)^{\sigma}\right) dy,$$

where

$$\beta_{r,k}(\sigma) = \frac{(-1)^{r-k}}{(r-2)!} \binom{r-2}{k-2} \prod_{i=1}^{r-1} (k-1-i\sigma), \quad k = 2, 3, \ldots, r.$$

By means of these operators and their modifications we can construct operators that satisfy Theorem 3.1 above for K-functionals $K(f, t; L_p(\chi_a^{\gamma_a}\chi_b^{\gamma_b})[a, b], W_p^r(\chi_a^{\gamma_a+r\lambda_a}\chi_b^{\gamma_b+r\lambda_b})[a, b])$ with $\lambda_a, \lambda_b \neq 1$. For example, by Propositions 3.9 and 6.2 in [16] (cf. Theorem 6.1 there) we establish

Theorem 4.1. *Let $r \in \mathbb{N}$ and $1 \leq p \leq \infty$, $(1-\lambda_a)(1-\nu_a) > 0$, $(1-\lambda_b)(1-\nu_b) > 0$. Let also $\kappa_a, \mu_a, \mu_b \notin \Gamma_{exc}(p)$ and $\kappa_b \in \Gamma_0(p)$. We set $w = \chi_a^{\kappa_a}\chi_b^{\kappa_b}$, $\varphi = \chi_a^{\lambda_a}\chi_b^{\lambda_b}$, $\tilde{w} = \chi_a^{\mu_a}\chi_b^{\mu_b}$, $\tilde{\varphi} = \chi_a^{\nu_a}\chi_b^{\nu_b}$ and*

$$\mathcal{A} = A_{i_1}(-\rho; a, b; \xi)A_{i_2}(-\rho_b; b, a; \xi)B_1(\sigma_b^{-1}; b, a; \xi)B_1(\sigma_a; a, b; \xi)A_0(\rho_a; a, b; \xi)$$
$$\mathcal{B} = A_{i'}(-\rho_a; a, b; \eta)B_1(\sigma_a^{-1}; a, b; \eta)B_1(\sigma_b; b, a; \eta)A_0(\rho_b; b, a; \eta)A_0(\rho; a, b; \eta),$$

where $\xi, \eta \in (a, b)$, $\rho < \mu_a + 1/p$, the integers i_1, i_2, i' are such that $\Gamma_{i_1}(p) \ni \mu_a$, $\Gamma_{i_2}(p) \ni \mu_b$, $\Gamma_{i'}(p) \ni \kappa_a$, and

$$\sigma_a = \frac{1-\nu_a}{1-\lambda_a}, \quad \sigma_b = \frac{1-\lambda_b}{1-\nu_b},$$
$$\rho_a = \kappa_a + \frac{1}{p} - \frac{\mu_a - \rho + 1/p}{\sigma_a}, \quad \rho_b = \mu_b + \frac{1}{p} - \frac{\kappa_b + 1/p}{\sigma_b}.$$

Then for $f \in L_p(w)[a, b]$ and $t > 0$ we have

$$K(f, t; L_p(w)[a, b], W_p^r(w\varphi^r)[a, b]) \sim K(\mathcal{A}f, t; L_p(\tilde{w})[a, b], W_p^r(\tilde{w}\tilde{\varphi}^r)[a, b])$$

and for $F \in L_p(\tilde{w})[a,b]$ *and* $t > 0$ *we have*

$$K(F,t;L_p(\tilde{w})[a,b],W_p^r(\tilde{w}\tilde{\varphi}^r)[a,b]) \sim K(\mathcal{B}F,t;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b]).$$

Remark 4.1. Interchanging a and b in the definition of $\mathcal{A}$ and $\mathcal{B}$ in the theorem above we get a similar relation between the K-functionals under the hypothesis that $\kappa_a \in \Gamma_0(p)$ and $\kappa_b, \mu_a, \mu_b \notin \Gamma_{exc}(p)$.

Theorem 4.1 and (2.1) imply directly a characterization of the considered K-functional by the ordinary modulus of smoothness but here we present another one given in [16, Theorem 6.2], which is simpler to state.

Theorem 4.2. *Let* $r \in \mathbb{N}$, $1 \le p \le \infty$ *and* $\lambda_a, \lambda_b \in (-\infty, 1)$. *For* $p < \infty$ *we assume that* $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$ *as at least one of them is in* $\Gamma_0(p)$, *and for* $p = \infty$ *we assume that* $\kappa_a = \kappa_b = 0$. *We set* $w = \chi_a^{\kappa_a}\chi_b^{\kappa_b}$, $\varphi = \chi_a^{\lambda_a}\chi_b^{\lambda_b}$ *and*

$$\mathcal{A} = B_1(\sigma_b;b,a;\xi)B_1(\sigma_a;a,b;\xi)A_0(\rho_b;b,a;\xi)A_0(\rho_a;a,b;\xi),$$

where $\xi \in (a,b)$ *and*

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_b = \frac{1}{1-\lambda_b}, \quad \rho_a = \kappa_a + \frac{\lambda_a}{p}, \quad \rho_b = \kappa_b + \frac{\lambda_b}{p}.$$

Then for $f \in L_p(w)[a,b]$ *and* $t > 0$ *we have*

$$K(f,t^r;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b]) \sim \omega_r(\mathcal{A}f,t)_{p[a,b]}.$$

The operators A and B defined in the beginning of this subsection can also be used when the weight exponent at the end s takes an exceptional value from $\Gamma_{exc}(p)$ but then they change the exponent into one that belongs to $\Gamma_{exc}(p)$ again. More precisely, the following assertion holds for the A-operators.

Proposition 4.1. *Let* $i, i' \in \mathbb{N}_0$, $r \in \mathbb{N}$, *as* $i, i' < r$, $1 \le p \le \infty$, $\gamma \in \Gamma_+(p)$, $\xi, \eta \in (a,b)$, *and* s *be one of the points* a *or* b *and* e *be the other one. We set* $w = \chi_s^{-i-1/p}\chi_e^{\gamma}$ *and* $\tilde{w} = \chi_s^{-i'-1/p}\chi_e^{\gamma}$. *Finally, let* ϕ *be measurable and non-negative on* (a,b). *Then we have*

$$K(f,t;L_p(w)[a,b],W_p^r(w\phi)[a,b]) \sim K(A_{i'}(i'-i;s,e;\xi)f,t;L_p(\tilde{w})[a,b],W_p^r(\tilde{w}\phi)[a,b])$$

and

$$K(F,t;L_p(\tilde{w})[a,b],W_p^r(\tilde{w}\phi)[a,b]) \\ \sim K(A_i(i-i';s,e;\eta)F,t;L_p(w)[a,b],W_p^r(w\phi)[a,b]).$$

P r o o f. Just similarly as in the proof of [16, Proposition 3.2] we verify that the operators $\mathcal{A} = A_{i'}(i'-i;s,e;\xi)$ and $\mathcal{B} = A_i(i-i';s,e;\eta)$ satisfy the hypotheses of Theorem 3.1 with $X_1 = L_p(w)[a,b]$, $X_2 = L_p(\tilde{w})[a,b]$, $Y_1 = W_p^r(w\phi)[a,b]$ and $Y_2 = W_p^r(\tilde{w}\phi)[a,b]$. In establishing properties a) and b) we also take into consideration that $\alpha_{r,i'+1}(i'-i) = 0$, $\alpha_{r,i+1}(i-i') = 0$ and hence Hardy's inequalities are applicable. □

If we separate the singularities of the weights w and φ beforehand, using (3.2), we can get a similar characterization of the K-functional with simpler transforms of the function but by a sum of two moduli ω_r. Moreover, the requirement that the exponent of the weight w on at least one of the ends of the interval is greater than $-1/p$ for $p < \infty$ is trivially satisfied and hence relaxed. In addition, Proposition 4.1 allows us to characterize the K-functional in the case $p = \infty$ not only for $\kappa_a, \kappa_b = 0$ but for all $\kappa_a, \kappa_b \in \Gamma_{exc}(\infty) = \{1-r, \dots, -1, 0\}$. Thus, (3.2), Theorem 4.2 (in the case $p < \infty$), and Proposition 4.1, [13, Theorem 5.4] (in the case $p = \infty$) yield the following relation (cf. [16, Theorem 7.1]).

Theorem 4.3. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$ and $\lambda_a, \lambda_b \in (-\infty, 1)$. For $p < \infty$ we assume that $\kappa_a, \kappa_b \notin \Gamma_{exc}(p)$, and for $p = \infty$ we assume that $\kappa_a, \kappa_b \in \Gamma_{exc}(\infty)$. We set $w = \chi_a^{\kappa_a}\chi_b^{\kappa_b}$, $\varphi = \chi_a^{\lambda_a}\chi_b^{\lambda_b}$ and*

$$\mathcal{A}_1 = B_1(\sigma_a;a,b_1;\xi_1)A_0(\rho_a;a,b_1;\xi_1),$$
$$\mathcal{A}_2 = B_1(\sigma_b;b,a_1;\xi_2)A_0(\rho_b;b,a_1;\xi_2),$$

where $a < a_1 < b_1 < b$, $\xi_1 \in (a,b_1)$, $\xi_2 \in (a_1,b)$ and

$$\sigma_a = \frac{1}{1-\lambda_a}, \quad \sigma_b = \frac{1}{1-\lambda_b}, \quad \rho_a = \kappa_a + \frac{\lambda_a}{p}, \quad \rho_b = \kappa_b + \frac{\lambda_b}{p}.$$

Then for $f \in L_p(w)[a,b]$ and $t > 0$ we have

$$K(f,t^r;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b]) \sim \omega_r(\mathcal{A}_1 f,t)_{p[a,b_1]} + \omega_r(\mathcal{A}_2 f,t)_{p[a_1,b]}.$$

Remark 4.2. The K-functional $K(f,t^r;L_p(w)[a,b],W_p^r(w\varphi^r)[a,b])$ for $\kappa_a,\kappa_b\in\Gamma_{exc}(p)$, $p<\infty$, and $\kappa_a,\kappa_b\notin\Gamma_{exc}(\infty)$, $p=\infty$, can also be characterized in a similar way but that involves new elements. Some initial comments on that are given in [15, Sections 3 and 4].

Results similar to those given in Theorems 4.1–4.3 are valid in the cases when one or both of the expressions $(1-\lambda_a)(1-\nu_a)$, $(1-\lambda_b)(1-\nu_b)$ is negative and/or the interval is (semi-)infinite (see [16]).

4.2. Exponential change of the variable. K-functionals with $\lambda_a = 1$ and/or $\lambda_\infty = 1$ are related to the approximation error of the Post-Widder, Gamma and Baskakov operators (see [17] and the references cited there, and also [31]). In [18] we show that

$$\|w(f-P_{1/t}f)\|_{p[0,\infty)}\sim\|w(f-G_{1/t}f)\|_{p[0,\infty)}\\ \sim K(f,t;L_p(w)[0,\infty),W_p^2(w\chi_0^2)[0,\infty)),$$

where $P_{1/t}$ and $G_{1/t}$ denote respectively the Post-Widder and the Gamma operators, $f\in L_p(w)[0,\infty)$, $w(x)=x^{\gamma_0}(1+x)^{\gamma_\infty-\gamma_0}$ with arbitrary $\gamma_0,\gamma_\infty\in\mathcal{R}$, and $1\le p\le\infty$ (the case $\gamma_0=\gamma_\infty$ was considered in [17]).

Following the ideas of the previous subsection for $r\in\mathcal{N}$, $\gamma\in\mathcal{R}$, $F\in L_{1,loc}(\mathcal{R})$, $f\in L_{1,loc}[a,\infty)$ and $x\in\mathcal{R}$ we define the operators

$$(A_\gamma F)(x)=e^{(\gamma+1/p)x}F(x)\\ +\sum_{k=1}^{r}(-1)^k\binom{r}{k}\frac{(\gamma+1/p)^k}{(k-1)!}\int_0^x(x-y)^{k-1}e^{(\gamma+1/p)y}F(y)\,dy,$$

$$(Bf)(x)=f(a+e^x)+\sum_{i=1}^{r-1}\frac{s(r,r-i)}{(i-1)!}\int_0^x(x-y)^{i-1}f(a+e^y)\,dy,$$

where $s(r,k)$ are the Stirling numbers of the first kind defined by

$$x(x-1)\dots(x-r+1)=\sum_{k=0}^{r}s(r,k)\,x^k$$

for $k=0,1,\dots,r$ and $s(r,k)=0$ for $k>r$. Then the following one-term characterization is valid [14, 19].

Theorem 4.4. *Let* $r\in\mathcal{N}$, $1\le p\le\infty$, $0<t\le t_0$, $\gamma\in\mathcal{R}$ *and* $f\in L_p(\chi_a^\gamma)[a,\infty)$.

a) If $\gamma \notin \Gamma_{exc}(p)$, then

$$K(f,t^r;L_p(\chi_a^\gamma)[a,\infty),W_p^r(\chi_a^{\gamma+r})[a,\infty)) \sim \omega_r(A_\gamma Bf,t)_{p(\mathcal{R})}.$$

b) If $\gamma \in \Gamma_{exc}(p)$, then

$$K(f,t^r;L_p(\chi_a^\gamma)[a,\infty),W_p^r(\chi_a^{\gamma+r})[a,\infty)) \sim \omega_r(B(\chi_a^{\gamma+1/p}f),t)_{p(\mathcal{R})}.$$

By means of the method of 3.1 the operators A_γ and B in the above theorem can be further simplified if we use two fixed step moduli of different order. To treat the more general weight $w(x) = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ with $\gamma_a, \gamma_\infty \in \mathcal{R}$ in some cases we shall also apply (3.2), which increases the number of fixed step moduli to four. For $r \in \mathcal{N}$, $i,j \in \mathcal{N}_0$, $j \le r$, distinct points $x_0,\dots,x_r \in (a,\infty)$ and a weight $\bar{w}$ we define the linear operator $\mathcal{A}_{i,j-1}(\bar{w}) : L_{1,loc}[a,\infty) \to L_{1,loc}(\mathcal{R})$ by

$$\mathcal{A}_{i,j-1}(\bar{w})f = (\bar{w}(f-\mathcal{L}_{i,j-1}f))\circ\mathcal{E},$$

where $\mathcal{E}(x) = e^x$ and

$$(\mathcal{L}_{i,j-1}f)(x) = \sum_{n=i}^{j-1}\frac{1}{n!}\left(\sum_{\ell=1}^{r}\frac{\Phi_\ell^{(n+1)}(a)}{\Phi_\ell(x_\ell)}\int_{x_0}^{x_\ell}f(y)\,dy\right)(x-a)^n,$$

$$\Phi_\ell(x) = \prod_{\substack{m=0\\ m\ne\ell}}^{r}(x-x_m), \quad \ell = 1,\dots,r.$$

We have the following characterization [18, Theorem 1.2].

Theorem 4.5. *Let $r \in \mathcal{N}$, $i,j \in \mathcal{N}_0$, $i,j \le r$, $1 \le p \le \infty$ and $t_0 > 0$. Let also $w = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ with $\gamma_a \in \Gamma_i(p)$, $\gamma_\infty \in \Gamma_j(p)$. Then for every $f \in L_p(w)[a,\infty)$ and $0 < t \le t_0$ there holds*

$$K(f,t^r;L_p(w)[a,\infty),W_p^r(w\chi_a^r)[a,\infty)) \sim \omega_r(\mathcal{A}_{i,j-1}(\chi_a^{1/p}w)f,t)_{p(\mathcal{R})} + t^r\,\|\mathcal{A}_{i,j-1}(\chi_a^{1/p}w)f\|_{p(\mathcal{R})}.$$

P r o o f. We shall show that the operator $\mathcal{A} = \mathcal{A}_{i,j-1}(\chi_a^{1/p}w)$ satisfies the hypotheses of Theorem 3.1 with $X_1 = L_p(w)[a,\infty)$, $Y_1 = W_p^r(w\chi_a^r)[a,\infty)$ as $|g|_{Y_1} = \|w\chi_a^r g^{(r)}\|_{p[a,\infty)}$, $X_2 = L_p(\mathcal{R})$, $Y_2 = W_p^r(\mathcal{R})$ as $|G|_{Y_2} = \|G\|_{p(\mathcal{R})} + \|G^{(r)}\|_{p(\mathcal{R})}$ and $\mathcal{B} : X_2 \to X_1$, defined by $\mathcal{B}F = \chi_a^{-1/p}w^{-1}(F\circ\log)$. Since $\mathcal{L}_{i,j-1}$:

$X_1 \to X_2$ is bounded we verify that $\mathcal{A}$ and $\mathcal{B}$ satisfy respectively conditions a) and c) of Theorem 3.1 just by a change of the variable. In [18, Proposition 4.3 and 4.4.e] we establish the inequalities

$$\|w\chi_a^k(g - \mathcal{L}_{i,j-1}g)^{(k)}\|_{p[a,\infty)} \le c\,\|w\chi_a^r g^{(r)}\|_{p[a,\infty)}, \quad k = 0,\dots,r.$$

provided that $g \in W_p^r(w\chi_a^r)[a,\infty)$, $\gamma_0 \in \Gamma_i(p)$, $\gamma_\infty \in \Gamma_j(p)$. Hence condition b) of Theorem 3.1 follows. Similarly, by the well-known inequalities

$$\|G^{(k)}\|_{p(\mathcal{R})} \le c\left(\|G\|_{p(\mathcal{R})} + \|G^{(r)}\|_{p(\mathcal{R})}\right), \quad k = 0,\dots,r,$$

we get d). Finally, we directly verify that $f - \mathcal{B}\mathcal{A}f = \mathcal{L}_{i,j-1}f \in \pi_{r-1} \cap Y_1$ for any $f \in X_1$, which implies e), and since $\mathcal{L}_{i,j-1}$ preserves the polynomials of the form $c_i\chi_a^i + \cdots + c_{j-1}\chi_a^{j-1}$ we have $\mathcal{A}\mathcal{B}F = F$ for any $F \in \mathcal{A}(X_1)$, which implies f) for $F \in \mathcal{A}(X_1)$.

Now, Theorem 3.1 in view of Remark 3.1 yields

$$K(f, t^r; L_p(w)[a,\infty), W_p^r(w\chi_a^r)[a,\infty)) \sim \inf\left\{\|\mathcal{A}f - G\|_{p(\mathcal{R})} + t^r\left(\|G\|_{p(\mathcal{R})} + \|G^{(r)}\|_{p(\mathcal{R})}\right) : G \in W_p^r(\mathcal{R})\right\}.$$

To complete the proof we just need to observe that for $F \in L_p(\mathcal{R})$, $1 \le p \le \infty$, and $0 < t \le t_0$ there holds (cf. [17, Lemma 5.2])

$$\inf\left\{\|F - G\|_{p(\mathcal{R})} + t^r\left(\|G^{(\ell)}\|_{p(\mathcal{R})} + \|G^{(r)}\|_{p(\mathcal{R})}\right) : G \in W_p^r(\mathcal{R})\right\} \sim \omega_r(F,t)_{p(\mathcal{R})} + t^{r-\ell}\omega_\ell(F,t)_{p(\mathcal{R})}, \quad \ell = 0,\dots,r-1.$$

□

Let us explicitly note that for $j \le i$ we have $\mathcal{A}_{i,j-1}(\chi_a^{1/p}w)f = (\chi_a^{1/p}wf) \circ \mathcal{E}$.

Similarly, the following assertion can be established

Theorem 4.6 [18, Theorem 1.3]. *Let $r \in \mathcal{N}$, $1 \le p \le \infty$ and $b, t_0 > 0$. Let also $w = \chi_a^{\gamma_a}\chi_{a-1}^{\gamma_\infty-\gamma_a}$ with $\gamma_a, \gamma_\infty \in \mathcal{R}$ and the integers i, j be determined by $\Gamma_i(p) \cup \{1-i-1/p\} \ni \gamma_a$, $\Gamma_j(p) \cup \{-j-1/p\} \ni \gamma_\infty$. We set $\ell_a = 1$ if $\gamma_a \in \Gamma_{exc}(p)$, and $\ell_a = 0$ otherwise. We set $\ell_\infty = 1$ if $\gamma_\infty \in \Gamma_{exc}(p)$, and $\ell_\infty = 0$ otherwise. Let the integers i', j' be such that $0 \le i' \le i - \ell_0$ and $j + \ell_\infty \le j' \le r$. Then for* *every* $f \in L_p(w)[a,\infty)$ *and* $0 < t \le t_0$ *there holds*

$$\begin{aligned} &K(f,t^r;L_p(w)[a,\infty),W_p^r(w\chi_a^r)[a,\infty)) \\ &\quad\sim \omega_r(\mathcal{A}_{i,j'-1}(\chi_a^{\gamma_a+1/p})f,t)_{p(-\infty,b]} + t^{r-\ell_a}\omega_{\ell_a}(\mathcal{A}_{i,j'-1}(\chi_a^{\gamma_a+1/p})f,t)_{p(-\infty,b]} \\ &\quad+ \omega_r(\mathcal{A}_{i',j-1}(\chi_a^{\gamma_\infty+1/p})f,t)_{p[-b,\infty)} + t^{r-\ell_\infty}\omega_{\ell_\infty}(\mathcal{A}_{i',j-1}(\chi_a^{\gamma_\infty+1/p})f,t)_{p[-b,\infty)}. \end{aligned}$$

Similar characterizations hold for the K-functionals $K(f,t;L_p(\chi_a^{\gamma_a})[a,b], W_p^r(\chi_a^{\gamma_a+r})[a,b])$ and $K(f,t;L_p(\chi_a^{\gamma_\infty})[a+1,\infty), W_p^r(\chi_a^{\gamma_\infty+r})[a+1,\infty))$.

4.3. A *K*-functional associated with the best approximation by trigonometric polynomials. Let $L_{p,2\pi}$ denote the set of the 2π-periodic functions in L_p. The best trigonometric approximation of a function $f \in L_{p,2\pi}$ is given by

$$E_n^T(f)_p = \inf_{g\in T_n} \|f-g\|_{p[-\pi,\pi]},$$

where T_n is the set of trigonometric polynomials of degree at most $n \in \mathbb{N}_0$. As it is known the rate of best approximation by trigonometric polynomials can be estimated by the periodic modulus of smoothness as follows (see e.g. [8, Ch. 7]):

$$\begin{aligned} E_n^T(f)_p &\le c\,\omega_r(f,n^{-1})_{p,2\pi}, \quad n\in\mathbb{N}, \\ \omega_r(f,t)_{p,2\pi} &\le c\,t^r \sum_{0\le k\le 1/t} (k+1)^{r-1}E_k^T(f)_p, \quad 0<t\le t_0. \end{aligned} \tag{4.2}$$

However, $\omega_r(f,t)_{p,2\pi} \equiv 0$ iff $f \in T_0$, whereas $E_n^T(f)_p = 0$ for any $f \in T_n$ and thus the direct estimate (4.2) contains a gap. This discrepancy can be overcome by defining another periodic modulus which is zero iff f is trigonometric polynomial of a given degree.

Let Π_n denote the set of the *algebraic* polynomials of degree $n \in \mathbb{N}_0$. For $r \in \mathbb{N}$ let us define the linear operator $\mathcal{A}_{r-1} : L_{p,2\pi} \to L_{p,2\pi} + \Pi_{2r-2}$ by

$$\mathcal{A}_{r-1}(f,x) = f(x) + \sum_{j=1}^{r-1} \frac{a_{r-1,j}}{(2j-1)!} \int_0^x (x-t)^{2j-1} f(t)\,dt,$$

where $a_{r-1,j}$ are given by the Stirling numbers of the first kind with

$$a_{r-1,j} = \sum_{k=1}^{2r-2j-1} (-1)^{r-j-k} s(r,k)\,s(r,2r-2j-k).$$

The first author introduced in [12] the following periodic modulus of smoothness:

$$\omega_r^T(f,t)_{p,2\pi} = \sup_{0<h\le t} \|\Delta_h^{2r-1}\mathcal{A}_{r-1}f\|_{p[-\pi,\pi]}. \tag{4.3}$$

Let us note that although $\mathcal{A}_{r-1}f$ is not generally a 2π-periodic function for $f \in L_{p,2\pi}$, its finite difference $\Delta_h^{2r-1}\mathcal{A}_{r-1}f$ is. It was established in [12] that

$$\begin{aligned} E_n^T(f)_p &\le c\,\omega_r^T(f,1/n)_{p,2\pi}, \quad n \ge r-1, \\ \omega_r^T(f,t)_{p,2\pi} &\le c\,t^{2r-1} \sum_{r-1\le k\le 1/t} (k+1)^{2r-2} E_k^T(f)_p, \quad 0 < t \le 1/r, \end{aligned} \tag{4.4}$$

as $\omega_r^T(f,t)_{p,2\pi} \equiv 0$ iff $f \in T_{r-1}$. Let us note that (4.4) for $n = r-1$ is a trigonometric analogue of Whitney's theorem.

A substantial element of the proof of (4.4) is the following relation between K-functionals, which is established by the method given in Theorem 3.1:

$$\begin{aligned} K_{r,\ell}^T(f,t)_p &= \inf\{\|f-g\|_{p[-\pi,\pi]} + t\|\widetilde{D}_r D^\ell g\|_{p[-\pi,\pi]} : g \in W_{p,2\pi}^{2r+\ell-1}\} \\ &\sim K(\mathcal{A}_{r-1}f, t; L_{p,2\pi} + \Pi_{2r-2}, W_{p,2\pi}^{2r+\ell-1}), \quad \ell = 0,1,\dots, \end{aligned}$$

with $Dg = g'$, $\widetilde{D}_r g = (D^2+(r-1)^2)\cdots(D^2+1)Dg$ and $W_{p,2\pi}^m = \{g \in AC_{loc}^m(\mathcal{R}) : D^m g \in L_{p,2\pi}\}$. Let us recall that $\widetilde{D}_r g = 0$ iff $g \in T_{r-1}$ and hence $K_{r,0}^T(f,t)_p \equiv 0$ iff $g \in T_{r-1}$.

Another modulus which is equivalent to zero for the trigonometric polynomials up to a given degree was considered by A.G. Babenko, N.I. Chernykh and V.T. Shevaldin. Through it they proved an upper estimate just like (4.2) for $p = 2$ and $r \in \mathcal{N}$ in [1], and Shevaldin [30] proved it for $p = \infty$ and $r = 2$.

4.4. The K-functional associated with the approximation error of the Kantorovich and the Durrmeyer operators.

Theorem 3.1 can be also applied for characterizing K-functionals with the second term generated by a linear differential operator of the form $P(D)$, where P is a polynomial with varying coefficients. In such cases the application is more complicated but the arising problems can be overcome, for example, by a varying sets technique. In order to demonstrate the approach let us consider the K-functional associated with the approximation error of the Kantorovich and the Durrmeyer operators.

Consider the space $L_p[0,1]$, $1 \le p \le \infty$, as for $p = \infty$ we identify $L_\infty[0,1]$ in this subsection with $C[0,1]$. Let $\tilde{Y}_p$ be $C^2[0,1]$ equipped with the semi-norm

$|g|_{\tilde{Y}_p} = \|(\phi g')'\|_{p[0,1]}$, where $\phi(x) = x(1-x)$. In this case the differential operator is $\phi D^2 + \phi' D$. As it was shown by Chen, Ditzian and Ivanov [7] for the Durrmeyer operator M_n and by Gonska and Zhou [20] for the Kantorovich operator K_n, we have for $1 \le p \le \infty$ the equivalence

$$\|f - K_n f\|_{p[0,1]} \sim \|f - M_n f\|_{p[0,1]} \sim K(f, 1/n; L_p[0,1], \tilde{Y}_p). \tag{4.5}$$

Further, Gonska and Zhou [20] proved for $f \in L_p[0,1]$, $1 < p \le \infty$ that

$$K(f, t^2; L_p[0,1], \tilde{Y}_p) \sim \omega^2_{\sqrt{\phi}}(f, t)_{p[0,1]} + \omega_1(f, t^2)_{p[0,1]},$$

where $\omega^2_{\sqrt{\phi}}$ is given by (2.3) with $w \equiv 1$. The above equivalence is not valid in the case $p = 1$. For the characterization of the K-functional in (4.5) for $p = 1$ the second author [23] used the scheme

$$\begin{aligned} K(f, t; L_1[0,1], \tilde{Y}_1) &= K(f, t; L_1[0,1], Z_1) \sim K(\mathcal{A}f, t; L_1[0,1], Z_2) \\ &\sim K(\mathcal{A}f, t; L_1[0,1], W_1^2(\phi)[0,1]) + t\,\omega_1(f, 1)_{1[0,1]}, \end{aligned} \tag{4.6}$$

where the operator $\mathcal{A}$ is given by

$$(\mathcal{A}f)(x) = f(x) + \int_{1/2}^{x} \left(\frac{x}{y^2} - \frac{1-x}{(1-y)^2} \right) f(y)\, dy.$$

Theorem 3.1 is applied in (4.6) with $X_1 = X_2 = L_1[0,1]$, $Y_1 = Z_1$ and $Y_2 = Z_2$, where

$$Z_1 = \left\{ f \in C^2[0,1] : f'(0) = 0,\ f'(1) = 0 \right\},$$

$$Z_2 = \left\{ f \in C^2[0,1] : f(0) = 2\int_0^{1/2} f(y)\, dy,\ f(1) = 2\int_{1/2}^{1} f(y)\, dy \right\}$$

and the semi-norms in Z_1 and Z_2 are given by $\|(\phi g')'\|_{1[0,1]}$ and $\|\phi g''\|_{1[0,1]}$ respectively. Note that Theorem 3.1 cannot be applied directly with $\mathcal{A}$ and the subspaces $Y_1 = \tilde{Y}_1$, $Y_2 = W_1^2(\phi)[0,1]$ because items b) and d) (with $\mathcal{B} = \mathcal{A}^{-1}$) are not fulfilled. Moreover, $\mathcal{A}(Z_1) = Z_2$ but $\mathcal{A}(\tilde{Y}_1) \ne W_1^2(\phi)[0,1]$. Using (4.5), (4.6) and Theorem 4.2 we get for every $n \in \mathbb{N}$ and every $f \in L_1[0,1]$

$$\|f - P_n f\|_{1[0,1]} \sim \|f - M_n f\|_{1[0,1]} \sim \omega_2(\tilde{A}\mathcal{A}f, n^{-1/2})_{1[0,1]} + n^{-1}\omega_1(f,1)_{1[0,1]},$$

where $\tilde{A}$ stays for the operator $\mathcal{A}$ from Theorem 4.2 with $r = 2$, $p = 1$, $a = 0$, $b = 1$, $\lambda_0 = \lambda_1 = 1/2$, $\kappa_0 = \kappa_1 = 0$, $\xi = 1/2$.

Following the approach sketched in this subsection Zapryanova [32] characterized the K-functional related to the $L_p[0,1]$, $1 \le p \le 2$, error of the algebraic version of the integral Jackson operator.

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