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Simultaneous approximation by Bernstein polynomials with integer coefficients

Borislav R. Draganov

Abstract

We prove that several forms of the Bernstein polynomials with integer coefficients possess the property of simultaneous approximation, that is, they approximate not only the function but also its derivatives. We establish direct estimates of the error of that approximation in uniform norm by means of moduli of smoothness. Moreover, we show that the sufficient conditions under which those estimates hold are also necessary.

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1 Main results

The Bernstein operator or polynomial is defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

Here $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of the *positive* integers. It is known that if $f \in C[0,1]$, then

$$\lim_{n\to\infty} \|B_n f - f\| = 0,$$

where $\|\circ\|$ is the sup-norm on the interval $[0,1]$. A best possible estimate of that convergence can be given by the Ditzian-Totik modulus of smoothness $\omega_\varphi^2(f,t)$ of the second order with a varying step, controlled by the weight $\varphi(x) := \sqrt{x(1-x)}$, in the uniform norm on the interval $[0,1]$. It is defined by (see [4, Chapter 2, (2.1.2)])

$$\omega_\varphi^2(f,t) := \sup_{0<h\le t} \|\bar{\Delta}_{h\varphi}^2 f\|,$$

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where

$$\bar{\Delta}^2_{h\varphi(x)} f(x) := \begin{cases} f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x)), & x \pm h\varphi(x) \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

For all $f \in C[0,1]$ and $n \in \mathbb{N}$ there holds (see [3, Chapter 10, (7.3)], or [2, Theorem 6.1])

$$\|B_n f - f\| \leq c\,\omega^2_\varphi(f, n^{-1/2}). \tag{1.1}$$

Above and henceforward c denotes a positive constant, not necessarily the same at each occurrence, whose value is independent of f and n.

The estimate (1.1) is best possible in the sense that its converse also holds true (see [12] and [20], or [3, Chapter 10, (7.3)], or [2, Theorem 6.1])

$$\omega^2_\varphi(f, n^{-1/2}) \leq c\,\|B_n f - f\|.$$

The varying-step moduli are quite useful when the approximation is better near the endpoints of the interval. Such is the case of the Bernstein polynomials, which interpolate the function at 0 and 1. More importantly, these moduli (unlike the classical ones) allow better inverse theorems for the best algebraic approximation since they take into account the effect of the endpoints (see [4, Chapter 7] and [3, Chapter 8]). Instead of $\omega^2_\varphi(f,t)$ we can use the moduli defined in [9, 10], [7, 13, 14, 15, 19], or [6].

Kantorovich [11] (or e.g. [1, pp. 3–4], or [16, Chapter 2, Theorem 4.1]) introduced an integer modification of B_n. It is given by

$$\widetilde{B}_n(f)(x) := \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k}.$$

Above $[\alpha]$ denotes the largest integer that is less than or equal to the real α. L. Kantorovich showed that if $f \in C[0,1]$ is such that $f(0), f(1) \in \mathbb{Z}$, then

$$\lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0.$$

Clearly, the conditions $f(0), f(1) \in \mathbb{Z}$ are also necessary in order to have $\lim_{n\to\infty} \widetilde{B}_n(f)(0) = f(0)$ and $\lim_{n\to\infty} \widetilde{B}_n(f)(1) = f(1)$, respectively.

Following L. Kantorovich and applying (1.1), we get a direct estimate of the error of $\widetilde{B}_n$ for $f \in C[0,1]$ with $f(0), f(1) \in \mathbb{Z}$. For $x \in [0,1]$ and $n \in \mathbb{N}$ we have

$$
\begin{aligned}
|\widetilde{B}_n(f)(x) - f(x)| &\le |B_n f(x) - f(x)| \\
&\quad + \sum_{k=1}^{n-1} \left(f\left(\frac{k}{n}\right) \binom{n}{k} - \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] \right) x^k (1-x)^{n-k} \\
&\le \|B_n f - f\| + \sum_{k=1}^{n-1} x^k (1-x)^{n-k} \qquad (1.2) \\
&\le c\,\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n} \sum_{k=1}^{n-1} p_{n,k}(x) \\
&\le c\,\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}.
\end{aligned}
$$

We will show that the simultaneous approximation by $\widetilde{B}_n(f)$ satisfies a similar estimate. Before stating that result, let us note that another integer modification of $B_n f$ possesses actually better properties regarding simultaneous approximation. In it, instead of the integer part $[\alpha]$ we use the nearest integer $\langle \alpha \rangle$ to the real α. More precisely, if $\alpha \in \mathbb{R}$ is not the arithmetic mean of two consecutive integers, we set $\langle \alpha \rangle$ to be the integer at which the minimum $\min_{m \in \mathbb{Z}} |\alpha - m|$ is attained. When α is right in the middle between two consecutive integers, we can define $\langle \alpha \rangle$ to be either of them even without following a given rule. The results we will prove are valid regardless of our choice in this case.

We will denote that integer modification of the Bernstein polynomial by $\widehat{B}_n(f)$, that is, we set

$$
\widehat{B}_n(f)(x) := \sum_{k=0}^{n} \left\langle f\left(\frac{k}{n}\right) \binom{n}{k} \right\rangle x^k (1-x)^{n-k}
$$

for $f \in C[0,1]$ and $x \in [0,1]$.

An argument similar to (1.2) yields

$$
\|\widehat{B}_n(f) - f\| \le c\,\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{2n}
$$

for all $f \in C[0,1]$ with $f(0), f(1) \in \mathbb{Z}$ and all $n \in \mathbb{N}$.

Let us explicitly note that for any fixed $n \ge 2$ the operator $\widetilde{B}_n : C[0,1] \to C[0,1]$ is not bounded in the sense that there does *not* exist a constant M such that

$$
\|\widetilde{B}_n f\| \le M\,\|f\| \quad \forall\, f \in C[0,1].
$$

Therefore we cannot drop the quantity $1/n$ on the right-hand side of the estimate (1.2), or replace it with $c\,\|f\|\,n^{-1}$. That operator is not continuous either. On the other hand, $\widehat{B}_n$ is bounded but not continuous. Both operators are not linear. To emphasize the latter we write $\widetilde{B}_n(f)$ and $\widehat{B}_n(f)$, not $\widetilde{B}_n f$ and $\widehat{B}_n f$.

Recently, we characterized the rate of the simultaneous approximation by the Bernstein operator with Jacobi weights in L_p-norm, $1 < p \le \infty$ (see [5]). In

particular, we showed in [5, Corollary 1.6] (with $r = 1$) that for all $f \in C^s[0,1]$ and $n \in \mathbb{N}$ there holds
(1.3)

$$\|(B_n f)^{(s)} - f^{(s)}\| \le c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}), & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\|, & s \ge 2, \end{cases}$$

as, moreover, these estimates cannot be improved. Here $\omega_1(F,t)$ is the ordinary modulus of continuity in the uniform norm on the interval $[0,1]$, defined by

$$\omega_1(F,t) := \sup_{\substack{|x-y|\le t\\ x,y\in[0,1]}} |F(x) - F(y)|.$$

We will verify that the integer forms of the Bernstein polynomials $\widetilde{B}_n$ and $\widehat{B}_n$ satisfy similar direct inequalities. They are stated in the following two theorems.

Theorem 1.1. *Let $s \in \mathbb{N}$. Let $f \in C^s[0,1]$ be such that $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$ and $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2,\dots,s$. Let also there exist $n_0 \in \mathbb{N}$, $n_0 \ge s$, such that*

$$f\left(\frac{k}{n}\right) \ge f(0) + \frac{k}{n}\,f'(0), \quad k = 1,\dots,s,\ n \ge n_0,$$
$$f\left(\frac{k}{n}\right) \ge f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s,\dots,n-1,\ n \ge n_0.$$

Then for $n \ge n_0$ there holds

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| \le c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}) + \dfrac{1}{n}, & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\| + \dfrac{1}{n}, & s \ge 2. \end{cases}$$

The constant c is independent of f and n.

Remark 1.2. An analogous result holds for the integer form of the Bernstein operator defined by means of the ceiling function instead of the integer part. Then we assume that the reverse inequalities for $f(k/n)$ hold, that is,

$$f\left(\frac{k}{n}\right) \le f(0) + \frac{k}{n}\,f'(0), \quad k = 1,\dots,s,\ n \ge n_0,$$
$$f\left(\frac{k}{n}\right) \le f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s,\dots,n-1,\ n \ge n_0.$$

The proof is quite similar and we will omit it.

The estimates of the rate of convergence for $\widehat{B}_n$ are valid under *weaker* assumptions.

Theorem 1.3. *Let $s \in \mathbb{N}$. Let $f \in C^s[0,1]$ be such that $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$ and $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \dots, s$. Then for $n \geq 1$ there holds*

$$\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| \leq c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}) + \dfrac{1}{n}, & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\| + \dfrac{1}{n}, & s \geq 2. \end{cases}$$

The constant c is independent of f and n.

We will also show that the assumptions made in Theorems 1.1 and 1.3 are necessary in order to have uniform simultaneous approximation. Concerning the difference between the set of conditions on the derivatives for $s = 1$ and $s \geq 2$, let us note that $\widetilde{B}_n$ and $\widehat{B}_n$ preserve the polynomials of the form $p\,x + q$, where $p, q \in \mathbb{Z}$ (that is verified just as for the Bernstein operator). Therefore it is not surprising that there are not any restrictions on the values of the function and its first derivative at the endpoints except that they must be integers. However, the requirement that the derivatives of order 2 and higher must be equal to 0 at the endpoints is quite unexpected. Technically, it is related to the fact that $\left(\frac{k}{n}\right)^s \binom{n}{k} \in \mathbb{Z}$ for all k and n iff $s = 0$ or $s = 1$.

There is an extensive literature on the approximation of functions by polynomials with integer coefficients. A quite helpful introduction to the subject is the monograph [1] (see also [16, Chapter 2, § 4]). In particular, the extension of the classical results on simultaneous approximation by algebraic polynomials with real coefficients to the integer case is due to Gelfond [8] and Trigub [21, 22]. Martinez [17] considered approximation of the derivatives of smooth functions by means of integer forms of the Bernstein polynomials but the coefficients are replaced by their integer part *after* differentiating the Bernstein polynomial of the function.

Finally, let us note that the approximation by polynomials with integer coefficients is important because of their computer implementations.

2 Proof of the estimates of the rate of convergence

The integer modifications of the Bernstein polynomials $\widetilde{B}_n$ and $\widehat{B}_n$ are not linear. That is why the simplest way to estimate their rate of approximation is to consider their deviation from the linear operator B_n (see (1.2)). We will apply the same approach to estimate their rate of simultaneous approximation.

For $n \in \mathbb{N}$ and $k = 0, \dots, n$, we set

$$\tilde{b}_n(k) := \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] \binom{n}{k}^{-1}$$

and

$$\hat{b}_n(k) := \left\langle f\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle \binom{n}{k}^{-1}.$$

Then the operators $\widetilde{B}_n$ and $\widehat{B}_n$ can be written respectively in the form

$$\widehat{B}_n(f)(x) = \sum_{k=0}^{n} \tilde{b}_n(k)\, p_{n,k}(x)$$

and

$$\widehat{B}_n(f)(x) = \sum_{k=0}^{n} \hat{b}_n(k)\, p_{n,k}(x).$$

We will use the forward finite difference operator Δ_h with step h, defined by

$$\Delta_h f(x) := f(x+h) - f(x), \quad \Delta_h^s := \Delta_h(\Delta_h^{s-1}).$$

Then

$$\Delta_h^s f(x) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} f(x + (s-i)h), \quad x \in [0, 1 - sh]. \tag{2.1}$$

If $h = 1$, we will omit the subscript, writing $\Delta := \Delta_1$. Thus

$$\Delta^s \tilde{b}_n(k) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} \tilde{b}_n(k + s - i), \quad k = 0, \ldots, n-s; \tag{2.2}$$

and analogously for $\hat{b}_n$.

As is known, for $n \geq s$ we have (see [18], or [3, Chapter 10, (2.3)], or [4, p. 125]) that

$$(B_n f)^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \Delta_{1/n}^s f\left(\frac{k}{n}\right) p_{n-s,k}(x), \quad x \in [0,1]. \tag{2.3}$$

Similarly, for $n \geq s$ we have

$$(\widetilde{B}_n(f))^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \Delta^s \tilde{b}_n(k)\, p_{n-s,k}(x), \quad x \in [0,1], \tag{2.4}$$

and

$$(\widehat{B}_n(f))^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \Delta^s \hat{b}_n(k)\, p_{n-s,k}(x), \quad x \in [0,1]. \tag{2.5}$$

We proceed to the results that relate $\widetilde{B}_n$ and $\widehat{B}_n$ to B_n.

Theorem 2.1. *Let $s \in \mathbb{N}$. Let $f \in C^s[0,1]$ be such that $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$ and $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \dots, s$. Let also there exist $n_0 \in \mathbb{N}$, $n_0 \geq s$, such that*

$$f\left(\frac{k}{n}\right) \geq f(0) + \frac{k}{n} f'(0), \quad k = 1, \dots, s, \ n \geq n_0, \tag{2.6}$$

$$f\left(\frac{k}{n}\right) \geq f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n - s, \dots, n-1, \ n \geq n_0. \tag{2.7}$$

Then

$$\|(B_n f)^{(s)} - (\widetilde{B}_n(f))^{(s)}\| \leq c \left(\omega_1(f^{(s)}, n^{-1}) + \frac{1}{n} \right), \quad n \geq n_0.$$

The constant c is independent of f and n.

Remark 2.2. Certainly, it suffices to assume instead of the cumbersome (2.6)-(2.7) that there exists $\delta \in (0,1)$ such that

$$f(x) \geq f(0) + x\, f'(0), \quad x \in [0, \delta],$$
$$f(x) \geq f(1) - (1-x) f'(1), \quad x \in [1-\delta, 1].$$

However, it turns out that the conditions (2.6)-(2.7) are also necessary unlike the ones above (see Theorem 3.2).

Theorem 2.3. *Let $s \in \mathbb{N}$. Let $f \in C^s[0,1]$ be such that $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$ and $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \dots, s$. Then*

$$\|(B_n f)^{(s)} - (\widehat{B}_n(f))^{(s)}\| \leq c \left(\omega_1(f^{(s)}, n^{-1}) + \frac{1}{n} \right), \quad n \geq 1.$$

The constant c is independent of f and n.

Now, Theorems 1.1 and 1.3 follow directly from (1.3) and Theorems 2.1 and 2.3, respectively.

Let us establish Theorems 2.1 and 2.3.

Proof of Theorem 2.1. Let $n \geq n_0$. We make use of (2.3), (2.4), and the identities $\sum_{j=0}^{s} \binom{s}{j} = 2^s$ and $\sum_{k=0}^{n-s} p_{n-s,k}(x) \equiv 1$ to get

$$\left| (B_n f)^{(s)}(x) - (\widetilde{B}_n(f))^{(s)}(x) \right| \leq 2^s\, n^s \max_{0 \leq k \leq n} \left(f\left(\frac{k}{n}\right) - \tilde{b}_n(k) \right), \quad x \in [0,1]. \tag{2.8}$$

Note that $f(k/n) - \tilde{b}_n(k) \geq 0$, $k = 0, \dots, n$, because $[\alpha] \leq \alpha$.

We will estimate $f(k/n) - \tilde{b}_n(k)$ separately for $k \le s$, $s+1 \le k \le n-s-1$, and $k \ge n-s$. For the middle part, we simply use that if $n \ge 2s+2$, then

$$
\begin{aligned}
f\left(\frac{k}{n}\right) - \tilde{b}_n(k) &= \left(f\left(\frac{k}{n}\right)\binom{n}{k} - \left[f\left(\frac{k}{n}\right)\binom{n}{k}\right]\right)\binom{n}{k}^{-1} \\
&\le \binom{n}{s+1}^{-1} \le \frac{c}{n^{s+1}}, \quad k = s+1, \dots, n-s-1.
\end{aligned} \tag{2.9}
$$

Next, we will show that

$$
f\left(\frac{k}{n}\right) - \tilde{b}_n(k) \le \frac{c}{n^s}\,\omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s. \tag{2.10}
$$

We apply Taylor's formula, as we take into consideration that $f^{(i)}(0) = 0$ for $i = 2, \dots, s$, to arrive at

$$
\begin{aligned}
f\left(\frac{k}{n}\right) &= f(0) + \frac{k}{n}\, f'(0) \\
&\quad + \frac{1}{(s-1)!}\int_0^{k/n} \left(\frac{k}{n} - t\right)^{s-1} \left(f^{(s)}(t) - f^{(s)}(0)\right) dt.
\end{aligned} \tag{2.11}
$$

That implies

$$
\begin{aligned}
f\left(\frac{k}{n}\right) - \left(f(0) + \frac{k}{n}\, f'(0)\right) &\le \frac{1}{s!}\left(\frac{k}{n}\right)^s \omega_1\left(f^{(s)}, \frac{k}{n}\right) \\
&\le \frac{c}{n^s}\,\omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s.
\end{aligned} \tag{2.12}
$$

At the second estimate, we have taken into account the well-known property of the modulus of continuity

$$
\omega_1(F, rt) \le r\omega_1(F, t),
$$

where $r \in \mathbb{N}$.

On the other hand, (2.6) and

$$
f(0)\binom{n}{k} + f'(0)\,\frac{k}{n}\binom{n}{k} \in \mathbb{Z}, \tag{2.13}
$$

imply

$$
\left[f\left(\frac{k}{n}\right)\binom{n}{k}\right] \ge f(0)\binom{n}{k} + f'(0)\,\frac{k}{n}\binom{n}{k}, \quad k = 0, \dots, s.
$$

Consequently,

$$
\tilde{b}_n(k) \ge f(0) + \frac{k}{n}\, f'(0), \quad k = 0, \dots, s. \tag{2.14}
$$

Estimates (2.12) and (2.14) imply (2.10).

Finally, we observe that, by symmetry, (2.10) yields

$$f\left(\frac{k}{n}\right) - \tilde{b}_n(k) \le \frac{c}{n^s}\,\omega_1(f^{(s)}, n^{-1}), \quad k = n-s, \dots, n. \tag{2.15}$$

More precisely, with $\bar{f}(x) := f(1-x)$ and

$$\bar{\tilde{b}}_n(k) := \left[\bar{f}\left(\frac{k}{n}\right)\binom{n}{k}\right]\binom{n}{k}^{-1}$$

we have

$$\begin{aligned} \bar{f}\left(\frac{k}{n}\right) &= f\left(\frac{n-k}{n}\right), \\ \bar{\tilde{b}}_n(k) &= \tilde{b}_n(n-k), \\ \omega_1(\bar{f}^{(s)}, t) &= \omega_1(f^{(s)}, t). \end{aligned} \tag{2.16}$$

Note also that $\bar{f} \in C^s[0,1]$, $\bar{f}(0), \bar{f}'(0) \in \mathbb{Z}$, $\bar{f}^{(i)}(0) = 0$, $i = 2, \dots, s$, and for $n \ge n_0$ and $k = 1, \dots, s$ we have by (2.7)

$$\bar{f}\left(\frac{k}{n}\right) = f\left(\frac{n-k}{n}\right) \ge f(1) - \frac{k}{n}f'(1) = \bar{f}(0) + \frac{k}{n}\bar{f}'(0).$$

So, $\bar{f}$ satisfies the condition (2.6) and, by virtue of (2.10), we have

$$\bar{f}\left(\frac{k}{n}\right) - \bar{\tilde{b}}_n(k) \le \frac{c}{n^s}\,\omega_1(\bar{f}^{(s)}, n^{-1}), \quad k = 0, \dots, s.$$

As we take into account (2.16), we get (2.15).

Inequalities (2.8)-(2.10) and (2.15) imply the assertion of the theorem. □

We will use the following elementary lemma in the proof the theorem about $\widehat{B}_n$.

Lemma 2.4. *Let $m \in \mathbb{Z}$ and $\alpha, \omega \in \mathbb{R}$. If $|\alpha - m| \le \omega$, then $|\langle\alpha\rangle - m| \le 2\omega$.*

Proof. If $\omega < 1/2$, then $\langle\alpha\rangle = m$. If, on the other hand, $\omega \ge 1/2$, then

$$|\langle\alpha\rangle - m| \le |\langle\alpha\rangle - \alpha| + |\alpha - m| \le \frac{1}{2} + \omega \le 2\omega.$$

□

Proof of Theorem 2.3. We proceed similarly to the proof of the previous theorem. Since the assertion is trivial for $n < s$, we assume that $n \ge s$. We make use of (2.3) and (2.5) to get

$$\begin{aligned} &\left|(B_n f)^{(s)}(x) - (\widehat{B}_n(f))^{(s)}(x)\right| \\ &\qquad \le 2^s\, n^s \max_{0\le k\le n}\left|f\left(\frac{k}{n}\right) - \hat{b}_n(k)\right|, \quad x \in [0,1]. \end{aligned} \tag{2.17}$$

Again we estimate separately the terms $|f(k/n) - \hat{b}_n(k)|$ for $k \le s$, $s+1 \le k \le n-s-1$, and $k \ge n-s$. For the middle part, we have similarly to (2.9)

$$\left| f\left(\frac{k}{n}\right) - \hat{b}_n(k) \right| \le \frac{c}{n^{s+1}}, \quad k = s+1, \dots, n-s-1, \ n \ge 2s+2. \tag{2.18}$$

Next, we will show that

$$\left| f\left(\frac{k}{n}\right) - \hat{b}_n(k) \right| \le \frac{c}{n^s}\, \omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s. \tag{2.19}$$

By virtue of (2.11), we have

$$\left| f\left(\frac{k}{n}\right) - \left(f(0) + \frac{k}{n}\, f'(0) \right) \right| \le \frac{c}{n^s}\, \omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s. \tag{2.20}$$

That implies

$$\begin{aligned} &\left| f\left(\frac{k}{n}\right) \binom{n}{k} - \left(f(0)\binom{n}{k} + f'(0)\, \frac{k}{n} \binom{n}{k} \right) \right| \\ &\qquad\qquad \le \frac{c}{n^s} \binom{n}{k} \omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s. \end{aligned} \tag{2.21}$$

We apply Lemma 2.4 with

$$\begin{aligned} \alpha &= f\left(\frac{k}{n}\right) \binom{n}{k}, \\ m &= f(0)\binom{n}{k} + f'(0)\, \frac{k}{n} \binom{n}{k} \in \mathbb{Z}, \\ \omega &= \frac{c}{n^s} \binom{n}{k} \omega_1(f^{(s)}, n^{-1}), \end{aligned}$$

where the constant c is the one on the right-hand side of (2.21).

Thus we arrive at

$$\begin{aligned} &\left| \left\langle f\left(\frac{k}{n}\right) \binom{n}{k} \right\rangle - \left(f(0)\binom{n}{k} + f'(0)\, \frac{k}{n} \binom{n}{k} \right) \right| \\ &\qquad\qquad \le \frac{c}{n^s} \binom{n}{k} \omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s, \end{aligned}$$

and, consequently,

$$\left| \hat{b}_n(k) - \left(f(0) + \frac{k}{n}\, f'(0) \right) \right| \le \frac{c}{n^s}\, \omega_1(f^{(s)}, n^{-1}), \quad k = 0, \dots, s. \tag{2.22}$$

Estimates (2.20) and (2.22) yield (2.19).

Finally, we derive

$$\left| f\left(\frac{k}{n}\right) - \hat{b}_n(k) \right| \le \frac{c}{n^s}\, \omega_1(f^{(s)}, n^{-1}), \quad k = n-s, \dots, n. \tag{2.23}$$

from (2.19) by symmetry just as in the proof of (2.15) with $\bar{\bar{b}}_n(k)$ replaced with

$$\bar{\hat{b}}_n(k) := \left\langle \bar{f}\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle \binom{n}{k}^{-1}.$$

Inequalities (2.17)-(2.19) and (2.23) imply the assertion of the theorem. $\square$

3 Optimality of the assumptions in Theorems 1.1 and 1.3

We will establish the necessity of the assumptions made in Theorems 1.1 and 1.3. We begin with the operator $\widehat{B}_n$ since stronger results are valid for it.

First of all, let us note that if

$$\text{(3.1)} \qquad \lim_{n\to\infty} \|\widehat{B}_n(f) - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0,$$

then $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}$ for $i = 0, \dots, s$. Indeed, as is known, for any $g \in C^s[0,1]$ we have (see e.g. [3, Chapter 2, Theorem 5.6])

$$\|g^{(i)}\| \le c\big(\|g\| + \|g^{(s)}\|\big), \quad i = 1, \dots, s-1.$$

Therefore (3.1) implies

$$\text{(3.2)} \qquad \lim_{n\to\infty} \|(\widehat{B}_n(f))^{(i)} - f^{(i)}\| = 0, \quad i = 0, \dots, s;$$

hence $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}$ for $i = 0, \dots, s$. A similar result holds for $\widetilde{B}_n$.

Theorem 3.1. *Let $s \in \mathbb{N}$, $s \ge 2$, and $f \in C^s[0,1]$. If*

$$\text{(3.3)} \qquad \lim_{n\to\infty} \|\widehat{B}_n(f) - f\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0,$$

then $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \dots, s$.

Proof. It is sufficient to establish the theorem at the point $x = 0$; for $x = 1$ it follows by symmetry. We use induction on s.

Let $s = 2$. Relation (3.2), in particular, yields

$$\lim_{n\to\infty} (\widehat{B}_n(f))'(0) = f'(0),$$

that is (see (2.5) with $s = 1$),

$$\text{(3.4)} \qquad \lim_{n\to\infty} n\Delta\hat{b}_n(0) = f'(0).$$

Since $n\Delta\hat{b}_n(0) \in \mathbb{Z}$ for all n, (3.4) implies

$$n\Delta\hat{b}_n(0) = f'(0) \quad \text{for } n \text{ large enough;}$$

hence

$$\hat{b}_n(1) = \hat{b}_n(0) + \frac{1}{n} f'(0) = f(0) + \frac{1}{n} f'(0). \tag{3.5}$$

Similarly, from $\lim_{n\to\infty}(\widehat{B}_n(f))''(0) = f''(0)$ we derive

$$n(n-1)\Delta^2 \hat{b}_n(0) = f''(0) \quad \text{for } n \text{ large enough.} \tag{3.6}$$

By Taylor's formula, we have

$$f\left(\frac{2}{n}\right) = f(0) + \frac{2}{n} f'(0) + \frac{2}{n^2} f''(0) + \int_0^{2/n} \left(\frac{2}{n} - t\right) (f''(t) - f''(0))\, dt. \tag{3.7}$$

Next, we proceed similarly to the proof of Theorem 2.3. We multiply both sides of the above identity by $\binom{n}{2}$ and rearrange the terms to get

$$\begin{aligned} f\left(\frac{2}{n}\right)\binom{n}{2} - \left(f(0)\binom{n}{2} + (n-1)f'(0) + f''(0)\right) \\ = -\frac{1}{n} f''(0) + \binom{n}{2} \int_0^{2/n} \left(\frac{2}{n} - t\right) (f''(t) - f''(0))\, dt. \end{aligned} \tag{3.8}$$

Consequently,

$$\begin{aligned} &\left| f\left(\frac{2}{n}\right)\binom{n}{2} - \left(f(0)\binom{n}{2} + (n-1)f'(0) + f''(0)\right) \right| \\ &\qquad \le \frac{1}{n} |f''(0)| + \omega_1\left(f'', \frac{2}{n}\right), \end{aligned}$$

which shows that for large n we have

$$\left\langle f\left(\frac{2}{n}\right)\binom{n}{2} \right\rangle = f(0)\binom{n}{2} + (n-1)f'(0) + f''(0).$$

Therefore

$$\hat{b}_n(2) = f(0) + \frac{2}{n} f'(0) + \frac{2}{n(n-1)} f''(0) \quad \text{for } n \text{ large enough.} \tag{3.9}$$

Now, fixing some n large enough, we deduce from (3.5)-(3.9) that

$$\begin{aligned} f''(0) &= n(n-1)(\hat{b}_n(2) - 2\hat{b}_n(1) + \hat{b}_n(0)) \\ &= n(n-1)\left(f(0) + \frac{2}{n} f'(0) + \frac{2}{n(n-1)} f''(0) - 2\left(f(0) + \frac{1}{n} f'(0)\right) + f(0)\right) \\ &= 2f''(0); \end{aligned}$$

hence $f''(0) = 0$.

Let the assertion of the theorem hold for some $s-1$, $s \geq 3$. We will prove that then it holds for s too.

As we noted in the beginning of the section, (3.3) implies

$$\lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s-1)} - f^{(s-1)}\| = 0.$$

Hence, by virtue of the induction hypothesis, we have $f^{(i)}(0) = 0$ for $i = 2, \ldots, s-1$.

By Taylor's formula we have

$$\text{(3.10)} \quad f\left(\frac{k}{n}\right) = f(0) + \frac{k}{n} f'(0) + \left(\frac{k}{n}\right)^s \frac{f^{(s)}(0)}{s!} + \frac{1}{(s-1)!}\int_0^{k/n} \left(\frac{k}{n} - t\right)^{s-1} (f^{(s)}(t) - f^{(s)}(0))\, dt.$$

We multiply both sides by $\binom{n}{k}$. For $1 \leq k < s$ we derive the inequality

$$\begin{aligned}
&\left| f\left(\frac{k}{n}\right)\binom{n}{k} - \left(f(0)\binom{n}{k} + f'(0)\frac{k}{n}\binom{n}{k}\right)\right| \\
&\qquad \leq \binom{n}{k}\left(\frac{k}{n}\right)^s \frac{1}{s!}\left(|f^{(s)}(0)| + \omega_1\left(f^{(s)}, \frac{k}{n}\right)\right) \\
&\qquad \leq \frac{c}{n}\left(|f^{(s)}(0)| + \omega_1(f^{(s)}, n^{-1})\right).
\end{aligned}$$

Consequently, for large n we have

$$\left\langle f\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle = f(0)\binom{n}{k} + f'(0)\frac{k}{n}\binom{n}{k};$$

hence

$$\text{(3.11)} \qquad \hat{b}_n(k) = f(0) + \frac{k}{n} f'(0) \quad \text{for } 0 \leq k < s \text{ and large } n.$$

In order to calculate $\hat{b}_n(s)$, we observe that

$$\lim_{n\to\infty} \binom{n}{s}\left(\frac{s}{n}\right)^s = \frac{s^s}{s!}.$$

We proceed just as in this case $s = 2$: we multiply both sides of (3.10) by $\binom{n}{s}$ and rearrange the terms to arrive at

$$\begin{aligned}
&\left| f\left(\frac{s}{n}\right)\binom{n}{s} - \left(f(0)\binom{n}{s} + f'(0)\frac{s}{n}\binom{n}{s} + \frac{s^s}{(s!)^2} f^{(s)}(0)\right)\right| \\
&\qquad \leq \left(\frac{s^s}{s!} - \binom{n}{s}\left(\frac{s}{n}\right)^s\right)\frac{1}{s!}|f^{(s)}(0)| + \frac{1}{s!}\binom{n}{s}\left(\frac{s}{n}\right)^s \omega_1\left(f^{(s)}, \frac{s}{n}\right) \\
&\qquad \leq \frac{c}{n}|f^{(s)}(0)| + c\,\omega_1(f^{(s)}, n^{-1}).
\end{aligned}$$

Consequently, for large n

$$\left\langle f\left(\frac{s}{n}\right)\binom{n}{s}\right\rangle = f(0)\binom{n}{s} + f'(0)\frac{s}{n}\binom{n}{s} + \left\langle \frac{s^s}{(s!)^2}\, f^{(s)}(0)\right\rangle + r_{s,n},$$

where $r_{s,n} \in \{-1, 0, 1\}$. Consequently,

$$\hat{b}_n(s) = f(0) + \frac{s}{n}\, f'(0) + \left(\left\langle \frac{s^s}{(s!)^2}\, f^{(s)}(0)\right\rangle + r_{s,n}\right)\binom{n}{s}^{-1}. \tag{3.12}$$

Relations (3.11) and (3.12) yield

$$\Delta^s \hat{b}_n(0) = \left(\left\langle \frac{s^s}{(s!)^2}\, f^{(s)}(0)\right\rangle + r_{s,n}\right)\binom{n}{s}^{-1}. \tag{3.13}$$

On the other hand, since $\lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0$, and, in particular, $\lim_{n\to\infty}(\widehat{B}_n(f))^{(s)}(0) = f^{(s)}(0)$, we have that

$$\lim_{n\to\infty} \frac{n!}{(n-s)!}\Delta^s \hat{b}_n(0) = f^{(s)}(0).$$

Taking into account that

$$\frac{n!}{(n-s)!}\Delta^s \hat{b}_n(0) \in \mathbb{Z} \quad \forall n,$$

we deduce that for large n there holds

$$\frac{n!}{(n-s)!}\Delta^s \hat{b}_n(0) = f^{(s)}(0).$$

That, in combination with (3.13), yields

$$s!\left(\left\langle \frac{s^s}{(s!)^2}\, f^{(s)}(0)\right\rangle + r_{s,n}\right) = f^{(s)}(0) \quad \text{for } n \text{ large enough.} \tag{3.14}$$

First of all, this relation implies that the integer $f^{(s)}(0)$ is divisible by $s!$, i.e. $f^{(s)}(0) = s!\, m_s$ with some $m_s \in \mathbb{Z}$. Secondly, it implies that $r_{s,n}$ has one and the same value for large n; denote it by r_s. Thus (3.14) can be reduced to

$$\left\langle \frac{s^s}{s!}\, m_s \right\rangle + r_s = m_s.$$

Consequently,

$$|m_s|\left(\frac{s^s}{s!} - 1\right) \le \frac{3}{2}.$$

It remains to take into account that $s^s/s!$ increases on s; hence $s^s/s! \ge 9/2$ for $s \ge 3$, and then $|m_s| \le 3/7$, which is possible only if $m_s = 0$. Thus $f^{(s)}(0) = 0$. □

Necessary conditions for the simultaneous approximation by means of $\widetilde{B}_n$ are given in the following theorem.

Theorem 3.2. *Let $s \in \mathbb{N}$ and $f \in C^s[0,1]$. If*

$$\lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = 0, \tag{3.15}$$

then $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2,\ldots,s$, and there exists $n_0 \in \mathbb{N}$, $n_0 \ge s$, such that

$$\begin{aligned} & f\left(\frac{k}{n}\right) \ge f(0) + \frac{k}{n} f'(0), \quad k = 1,\ldots,s,\ n \ge n_0, \\ & f\left(\frac{k}{n}\right) \ge f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s,\ldots,n-1,\ n \ge n_0. \end{aligned} \tag{3.16}$$

Proof. It is sufficient to establish the theorem at the point $x = 0$; for $x = 1$ it follows by symmetry.

We argue as in the proof of the preceding theorem. However, here more efforts are required.

Using induction on s, we will prove that $f^{(i)}(0) = 0$, $i = 2,\ldots,s$ and

$$\tilde{b}_n(k) = f(0) + \frac{k}{n} f'(0), \quad k = 1,\ldots,s,\ n \ge n_0. \tag{3.17}$$

with some n_0. The latter implies directly the inequalities (3.16) because

$$f\left(\frac{k}{n}\right) \ge \left[f\left(\frac{k}{n}\right)\binom{n}{k}\right]\binom{n}{k}^{-1} = f(0) + \frac{k}{n} f'(0), \quad k = 1,\ldots,s,\ n \ge n_0.$$

As in the proof of Theorem 3.1, we deduce from

$$\lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = 0$$

that there exists $n_0 \in \mathbb{N}$, $n_0 \ge s$, such that

$$\frac{n!}{(n-i)!}\Delta^i \tilde{b}_n(0) = f^{(i)}(0), \quad i = 1,\ldots,s,\ n \ge n_0. \tag{3.18}$$

That directly yields (3.17) for $s = 1$ and the assertion of the theorem is verified for $s = 1$.

In order to complete the proof for larger s, we use that if $f \in C^s[0,1]$ and $\lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = 0$, then

$$\lim_{n\to\infty} \|(B_n f)^{(s)} - (\widetilde{B}_n(f))^{(s)}\| = 0;$$

hence

$$\lim_{n\to\infty} \left((B_n f)^{(s)}\left(\frac{y}{n}\right) - (\widetilde{B}_n(f))^{(s)}\left(\frac{y}{n}\right)\right) = 0, \quad y \in [0,1]. \tag{3.19}$$

By (2.1)-(2.4), after reordering the terms, we arrive at the identity

$$(3.20)\quad (B_n f)^{(s)}(x) - (\widetilde{B}_n(f))^{(s)}(x) = \frac{n!}{(n-s)!}\sum_{k=0}^{n-s}\sum_{j=k}^{k+s}(-1)^{s+j-k}\binom{s}{j-k}\left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j)\right)p_{n-s,k}(x).$$

We observe that, by virtue of (2.9), for $n \geq 3s+2$ and $x \in [0,1]$ there holds (cf. (2.8))

$$(3.21)\quad \left|\sum_{k=s+1}^{n-2s-1}\sum_{j=k}^{k+s}(-1)^{s+j-k}\binom{s}{j-k}\left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j)\right)p_{n-s,k}(x)\right| \leq \frac{c}{n^{s+1}}$$

and

$$(3.22)\quad \left|\sum_{k=1}^{s}\sum_{j=s+1}^{k+s}(-1)^{s+j-k}\binom{s}{j-k}\left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j)\right)p_{n-s,k}(x)\right| \leq \frac{c}{n^{s+1}}.$$

Next, we observe that if $n \geq 4s+1$, then $p_{n-s,k}(y/n) \leq c\,n^{-s-1}$ for all $y \in [0,1]$ and $k = n-2s, \dots, n-s$. Indeed, since in this case $(n-s)/2 \leq n-2s$, then for $k = n-2s, \dots, n-s$ there holds

$$\binom{n-s}{k} \leq \binom{n-s}{n-2s} = \binom{n-s}{s} \leq c\,n^s.$$

Next, we take into account that for $n \geq 4s+1$ and $k \geq n-2s$ we have $k \geq 2s+1$; hence

$$\frac{y^k}{n^k} \leq \frac{1}{n^{2s+1}}, \quad y \in [0,1].$$

These two relations along with the trivial estimate $(1-y/n)^{n-s-k} \leq 1$ imply that $p_{n-s,k}(y/n) \leq c\,n^{-s-1}$ for all $y \in [0,1]$ and $k = n-2s, \dots, n-s$, $n \geq 4s+1$.

Further, taking also into account that $0 \leq f(j/n) - \tilde{b}_n(j) \leq 1$ and arguing as in (2.9), we arrive at

$$(3.23)\quad \left|\sum_{k=n-2s}^{n-s}\sum_{j=k}^{k+s}(-1)^{s+j-k}\binom{s}{j-k}\left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j)\right)p_{n-s,k}\left(\frac{y}{n}\right)\right| \leq \frac{c}{n^{s+1}}, \quad y \in [0,1].$$

We subtract (3.21) and (3.22) with $x = y/n$, and (3.23) from (3.20) with $x = y/n$, reorder the terms and take into account (3.19) and $\tilde{b}_n(0) = f(0)$. Thus, for $y \in [0,1]$, we deduce that

$$(3.24)\quad \lim_{n\to\infty}\frac{n!}{(n-s)!}\sum_{j=1}^{s}(-1)^{s-j}\left(f\left(\frac{j}{n}\right) - \tilde{b}_n(j)\right)\sum_{k=0}^{j}(-1)^k\binom{s}{j-k}p_{n-s,k}\left(\frac{y}{n}\right) = 0.$$

We will evaluate that limit in another way. Clearly,

$$\lim_{n\to\infty}\sum_{k=0}^{j}(-1)^k\binom{s}{j-k}p_{n-s,k}\left(\frac{y}{n}\right)=\frac{1}{e^y}\sum_{k=0}^{j}(-1)^k\frac{y^k}{k!}\binom{s}{j-k}. \tag{3.25}$$

We proceed by induction on s. Relations (3.15) imply

$$\lim_{n\to\infty}\|(\widetilde{B}_n(f))^{(s-1)}-f^{(s-1)}\|=0.$$

Therefore, by virtue of the induction hypothesis, we have that $f^{(i)}(0)=0$, $i=2,\dots,s-1$, $s\geq 2$, and

$$\tilde{b}_n(j)=f(0)+\frac{j}{n}f'(0),\quad j=1,\dots,s-1,\ n\geq n_0. \tag{3.26}$$

Then Taylor's formula yields

$$f\left(\frac{j}{n}\right)=f(0)+\frac{j}{n}f'(0)+\frac{j^s}{n^s}\frac{f^{(s)}(0)}{s!}+o(n^{-s}),\quad j=1,\dots,s.$$

The relations (3.18) with $i=s$ and (3.26) imply

$$\tilde{b}_n(s)=f(0)+\frac{s}{n}f'(0)+\frac{(n-s)!}{n!}f^{(s)}(0),\quad n\geq n_0. \tag{3.27}$$

Therefore

$$\frac{n!}{(n-s)!}\left(f\left(\frac{j}{n}\right)-\tilde{b}_n(j)\right)=\frac{j^s}{s!}f^{(s)}(0)+o(1),\quad j=1,\dots,s-1,$$

and

$$\frac{n!}{(n-s)!}\left(f\left(\frac{s}{n}\right)-\tilde{b}_n(s)\right)=\left(\frac{s^s}{s!}-1\right)f^{(s)}(0)+o(1).$$

Now, if we substitute the last two relations into (3.24) and take into account (3.25), we arrive at

$$f^{(s)}(0)\sum_{k=0}^{s}\frac{(-1)^k y^k}{k!}\left(\binom{s}{k}-\sum_{j=k}^{s}(-1)^{s-j}\frac{j^s}{s!}\binom{s}{j-k}\right)=0,\quad y\in[0,1]$$

(actually the summand for $k=0$ is 0). Consequently, the coefficient of y^s is equal to zero, that is,

$$\frac{(-1)^s f^{(s)}(0)}{s!}\left(1-\frac{s^s}{s!}\right)=0.$$

Therefore $f^{(s)}(0)=0$ and then, by virtue of (3.27), $\tilde{b}_n(s)=f(0)+\frac{s}{n}f'(0)$. □

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Borislav R. Draganov
Dept. of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
Bulgaria
bdraganov@fmi.uni-sofia.bg

Inst. of Mathematics and Informatics
Bulgarian Academy of Sciences
bl. 8 Acad. G. Bonchev Str.
1113 Sofia
Bulgaria