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STRONG CONVERSE INEQUALITIES FOR THE WEIGHTED SIMULTANEOUS APPROXIMATION BY THE SZÁSZ-MIRAKJAN OPERATOR*

Borislav R. Draganov

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Abstract. We establish two-term strong converse estimates of the rate of weighted simultaneous approximation by the Szász-Mirakjan operator for smooth functions in the supremum norm on the non-negative semi-axis. We consider Jacobi-type weights. The estimates are stated in terms of appropriate moduli of smoothness or K-functionals.

1. Main results. The Szász-Mirakjan operator for a function $f(x)$ defined on $[0,\infty)$ is given by

$$S_n f(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) s_{n,k}(x), \quad s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \quad n \ge 1, \quad x \ge 0,$$

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as n is not necessarily an integer.

Let $C[0,\infty)$ denote the space of the continuous, not necessarily bounded, functions on $[0,\infty)$, and $L_\infty[0,\infty)$ be the space of the essentially bounded Lebesgue measurable function on $[0,\infty)$, equipped with the essential supremum norm $\|\circ\|$.

We will consider simultaneous approximation by the Szász-Mirakjan operator in the essential supremum norm on $[0,\infty)$ with weights of the form

$$w(x) = w(\gamma_0,\gamma_\infty;x) = \left(\frac{x}{1+x}\right)^{\gamma_0}(1+x)^{\gamma_\infty}. \tag{1.1}$$

Let $r \in \mathbb{N}_+$ and $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. We denote by $\mathbb{N}_+$ the set of the positive integers. In [8, Theorem 1.2] we proved the direct estimate

$$\|w(S_n f - f)^{(r)}\| \le c\,\widetilde{K}_r(f^{(r)}, n^{-1})_w$$

for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \ge 1$. Here and henceforward c stands for a positive constant (not necessarily the same at each occurrence), which is independent of the approximated function f and the degree of the operator n. The K-functional $\widetilde{K}_r(f^{(r)}, t)_w$ is defined by

$$\widetilde{K}_r(f^{(r)}, t)_w := \inf\Big\{\|w(f^{(r)} - g^{(r)})\| + t\|w\big(\widetilde{D}g\big)^{(r)}\| \\ : g \in AC^{r+1}[0,\infty),\ wg^{(r)}, w\big(\widetilde{D}g\big)^{(r)} \in L_\infty[0,\infty)\Big\},$$

where $\widetilde{D}g(x) := xg''(x)$, $AC^m[0,\infty)$ is the set of the functions which along with their derivatives up to order m are absolutely continuous on $[a,b]$ for every $[a,b] \subset [0,\infty)$.

In the present paper, we will establish the following converse inequality.

Theorem 1.1. *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0,\gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Then there exists $R \ge 1$ such that for all $f \in C[0,\infty)$ with $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $k, n \ge 1$ with $k \ge Rn$ there holds*

$$\widetilde{K}_r(f^{(r)}, n^{-1})_w \le c\frac{k}{n}\left(\|w(S_n f - f)^{(r)}\| + \|w(S_k f - f)^{(r)}\|\right).$$

In particular,

$$\widetilde{K}_r(f^{(r)}, n^{-1})_w \le c\left(\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\|\right).$$

The constant $c > 0$ is independent of f, k and n.

The rate of the simultaneous approximation by the Szász-Mirakjan operator can be estimated by simpler function characteristics—moduli of smoothness.

We will use the weighted Ditzian-Totik modulus of smoothness $\omega_\varphi^2(f,t)_w$ defined in [5, p. 56] with $\varphi(x) := \sqrt{x}$ and the weighted modulus of continuity

$$\omega(f,t)_w := \sup_{0<h\le t} \|w\overrightarrow{\Delta}_h f\|,$$

where

$$\overrightarrow{\Delta}_h f(x) := f(x+h) - f(x), \quad x \ge 0.$$

In [8, Theorem 1.1] it was established that

$$\|w(S_n f - f)^{(r)}\| \le c\left(\omega_\varphi^2(f^{(r)}, n^{-1/2})_w + \omega(f^{(r)}, n^{-1})_w\right), \quad n \ge n_0, \tag{1.2}$$

with some $n_0 \ge 1$ for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$ provided that $0 \le \gamma_0 < r$, whereas γ_∞ is arbitrary. Also, there was shown that the second term on the right above is redundant if $0 < \gamma_0 < r$ and $\gamma_\infty > 0$.

Here we will derive from Theorem 1.1 the following converse estimate.

Theorem 1.2. *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Then there exist $R, n_0 \ge 1$ such that for all $f \in C[0,\infty)$ with $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$ there hold*

$$\omega_\varphi^2(f^{(r)}, n^{-1/2})_w \le c\left(\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\|\right), \quad n \ge n_0,$$

and

$$\omega(f^{(r)}, n^{-1})_w \le c\left(\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\|\right), \quad n \ge 1.$$

The constant $c > 0$ is independent of f and n.

We say that the real-valued functions $A(f,n)$ and $B(f,n)$ are equivalent and write $A(f,n) \sim B(f,n)$ for f and n in specified domains iff there exists a positive constant c such that $c^{-1}B(f,n) \le A(f,n) \le c\,B(f,n)$ for all f and n in the specified domains.

Theorems 1.1 and 1.2, [8, Theorems 1.1 and 1.2], and properties of the K-functionals and moduli (see [5, Theorem 6.1.1]) imply the following equivalences.

Corollary 1.3. *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Then there exist $R, n_0 \ge 1$ such that for all $f \in C[0,\infty)$ with $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \ge n_0$ there hold*

$$\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\| \sim \widetilde{K}_r(f^{(r)}, n^{-1})_w$$

$$\sim \omega_\varphi^2(f^{(r)}, n^{-1/2})_w + \omega(f^{(r)}, n^{-1})_w.$$

In particular, the direct inequality (1.2) and Theorem 1.2 (or Corollary 1.3) readily imply a big O-characterization of the rate of the simultaneous approximation by the Szász-Mirakjan operator.

Corollary 1.4. *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Let also $f \in C[0,\infty)$ be such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and $0 < \alpha \le 1$. Then*

$$\|w(S_n f - f)^{(r)}\| = O(n^{-\alpha}) \iff \omega_\varphi^2(f^{(r)}, t)_w = O(t^{2\alpha}) \quad \text{and} \quad \omega(f^{(r)}, t)_w = O(t^\alpha).$$

The approximation of f' with $(S_n f)'$ is closely related to the approximation by means of the Szász-Mirakjan-Kantorovich operator. This operator is defined for functions $f(x)$, which are summable on every compact subinterval of $[0,\infty)$, by

$$\widetilde{S}_n f(x) := \sum_{k=0}^{\infty} s_{n,k}(x)\, n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)\, du, \quad x \ge 0.$$

We set

$$F(x) := \int_0^x f(u)\, du, \quad x \ge 0.$$

Then, by virtue of (2.8) below,

$$\widetilde{S}_n f(x) = (S_n F)'(x).$$

Now, Theorems 1.1 and 1.2 yield the following converse inequalities for the simultaneous approximation by the Szász-Mirakjan-Kantorovich operator in weighted L_∞-spaces.

Theorem 1.5. *Let $r \in \mathbb{N}_0$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r+1$ and $\gamma_\infty \ne r+1$. Then there exists $R \ge 1$ such that for all $f(x)$, which are summable on every compact subinterval of $[0,\infty)$, $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \ge 1$ there holds*

$$\widetilde{K}_{r+1}(f^{(r)}, n^{-1})_w \le c\left(\|w(\widetilde{S}_n f - f)^{(r)}\| + \|w(\widetilde{S}_{Rn} f - f)^{(r)}\|\right).$$

Theorem 1.6. *Let $r \in \mathbb{N}_0$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1), as $0 \le \gamma_0 < r+1$ and $\gamma_\infty \ne r+1$. Then there exist $R, n_0 \ge 1$ such that for all $f(x)$,* *which are summable on every compact subinterval of* $[0,\infty)$*,* $f \in AC_{loc}^{r-1}(0,\infty)$ *and* $wf^{(r)} \in L_\infty[0,\infty)$ *there hold*

$$\omega_\varphi^2(f^{(r)}, n^{-1/2})_w \le c\left(\|w(\widetilde{S}_n f - f)^{(r)}\| + \|w(\widetilde{S}_{Rn} f - f)^{(r)}\|\right), \quad n \ge n_0,$$

and

$$\omega(f^{(r)}, n^{-1})_w \le c\left(\|w(S_n f - f)^{(r)}\| + \|w(S_{Rn} f - f)^{(r)}\|\right), \quad n \ge 1.$$

The constant $c > 0$ *is independent of* f *and* n*.*

Here the assumption $f \in AC_{loc}^{r-1}(0,\infty)$ is to be ignored for $r = 0$. The unweighted case, that is $w = 1$, for $r = 0$ was considered in [10] in $L_p[0,\infty)$, $1 < p \le \infty$. Weaker converse results for $r = 0$, but for more general operators in some instances, were obtained earlier in [5, Theorems 9.3.2 and 10.1.3] and [14, 15].

The contents of the paper are organized as follows. In the next section we establish a Voronovskaya-type estimate and several Bernstein-type inequalities for the simultaneous approximation by the Szász-Mirakjan operator in weighted L_∞-norm. Then, in the last section, we apply them to verify Theorem 1.1 and by means of the method for proving converse inequalities, described in [4]. There we also give a proof of Theorem 1.2.

2. Basic assertions. We begin with several notations and known auxiliary results.

Let $AC_{loc}^m(0,\infty)$ denote the set of the functions which along with their derivatives up to order m are absolutely continuous on $[a,b]$ for every $[a,b] \subset (0,\infty)$.

We set $s_{n,k} := 0$ for $k < 0$. Direct computations yield the following two formulas for the derivatives of $s_{n,k}(x)$, $k \in \mathbb{N}_0$:

$$s'_{n,k}(x) = n(s_{n,k-1}(x) - s_{n,k}(x)) \tag{2.1}$$

and

$$s'_{n,k}(x) = \frac{1}{x}(k - nx)\, s_{n,k}(x). \tag{2.2}$$

For a sequence $\{a_k\}_{k\in\mathbb{Z}}$ we define $\Delta a_k := a_k - a_{k-1}$ and $\Delta^r a_k := \Delta(\Delta^{r-1} a_k)$. Set $s_k(n,x) := s_{n,k}(x)$. Then iterating (2.1), we get

$$s_{n,k}^{(r)}(x) = (-1)^r n^r \Delta^r s_k(n,x). \tag{2.3}$$

Likewise, using (2.2), we get by induction on r the formula (cf. [5, (9.4.9)])

$$s_{n,k}^{(r)}(x) = x^{-r} s_{n,k}(x) \sum_{0\le i\le r/2} (nx)^i \sum_{j=0}^{r-2i} d_{r,i,j}(k-nx)^j, \tag{2.4}$$

where $d_{r,i,j}$ are constants, whose value is independent of n and k.

For $\ell \in \mathbb{N}_0$ we set

$$T_{n,\ell}(x) := n^\ell S_n\left((\circ - x)^\ell\right)(x) = \sum_{k=0}^{\infty}(k-nx)^\ell s_{n,k}(x). \tag{2.5}$$

As is known (see [5, Lemma 9.5.5]), we have for $\ell \ge 1$

$$T_{n,\ell}(x) = \sum_{1\le\rho\le\ell/2} d_{\ell,\rho}(nx)^\rho,$$

where $d_{\ell,\rho}$ are constants, whose value is independent of n. We follow the convention that an empty sum is identically 0. In particular, we have (see e.g. [12, p. 94])

$$\begin{aligned} &T_{n,0}(x) = 1, \quad T_{n,1}(x) = 0, \quad T_{n,2}(x) = T_{n,3}(x) = nx,\\ &T_{n,4}(x) = 3(nx)^2 + nx. \end{aligned} \tag{2.6}$$

Identity (2.5) yields for $m \ge 1$

$$0 \le T_{n,2\ell}(x) \le c\begin{cases} nx, & nx \le 1,\\ (nx)^\ell, & nx \ge 1. \end{cases}$$

Then, by means of Cauchy's inequality and the identity $\sum_{k=0}^{\infty} s_{n,k}(x) \equiv 1$, we get

$$0 \le \sum_{k=0}^{\infty} |k-nx|^\ell s_{n,k}(x) \le \sqrt{T_{n,2\ell}(x)} \le c\begin{cases} 1, & nx \le 1,\\ (nx)^{\ell/2}, & nx \ge 1. \end{cases} \tag{2.7}$$

We will also use the quantities

$$T_{r,n,\ell}(x) := \sum_{k=0}^{\infty}(k-nx)^\ell s_{n,k}^{(r)}(x).$$

To recall, the forward finite difference of $f : [0,\infty) \to \mathbb{R}$ with step $h > 0$ is defined by $\overrightarrow{\Delta}_h f(x) := f(x+h) - f(x)$, $x \ge 0$. We have the following formula for its rth iterate, $\overrightarrow{\Delta}_h^r := \overrightarrow{\Delta}_h(\overrightarrow{\Delta}_h^{r-1})$,

$$\overrightarrow{\Delta}_h^r f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x + (r-i)h), \quad x \geq 0.$$

As is known (see [13] or [5, (9.4.3)])

$$(S_n f)^{(r)}(x) = n^r \sum_{k=0}^{\infty} \overrightarrow{\Delta}_{1/n}^r f\left(\frac{k}{n}\right) s_{n,k}(x), \quad x \geq 0. \tag{2.8}$$

In [8, Proposition 3.1] it was shown that if $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ is given by (1.1) with $0 \leq \gamma_0 < r$ and $\gamma_\infty \in \mathbb{R}$, then for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \geq 1$ there holds

$$\|w(S_n f)^{(r)}\| \leq c\,\|wf^{(r)}\|. \tag{2.9}$$

Next, we will establish a Voronovskaya-type inequality. A basic tool in its proof is the following formula.

Lemma 2.1. *Let $r \in \mathbb{N}_+$, $\gamma \in \mathbb{R}$ and $n \geq 1$. Let also $f \in C[0,\infty)$ be such that $\varphi^\gamma f \in L_\infty[1,\infty)$, $f \in AC_{loc}^{r+3}(0,\infty)$ and $\varphi^{2r+6} f^{(r+4)} \in L[0,1]$. Then*

$$\begin{aligned}
&\left(S_n f(x) - f(x) - \frac{1}{2n}\,\widetilde{D} f(x)\right)^{(r)} \\
&\quad = \frac{S(r+2,r)}{(r+1)(r+2)n^2}\, f^{(r+2)}(x) \\
&\quad + \left(\frac{(3r+2)x}{12n^2} + \frac{S(r+3,r)}{(r+1)(r+2)(r+3)n^3}\right) f^{(r+3)}(x) \\
&\quad + \frac{1}{(r+3)!} \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{r+3} f^{(r+4)}(u)\,du, \quad x > 0.
\end{aligned}$$

Here $S(m,r) := \dfrac{1}{r!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} (r-i)^m$ are the Stirling numbers of the second kind.

Proof. By [7, Proposition 2.1] with $p = 1$, $g = f$, $j = r+2, r+3$, $m = r+4$, $w_1 = \varphi^{2j-2}$ and $w_2 = \varphi^{2r+6}$ we get

$$\varphi^{2j-2} f^{(j)} \in L[0,1], \quad j = r+2, r+3. \tag{2.10}$$

Then (see e.g. [7, p. 106, (3.11)]) we have

$$\lim_{u \to 0+0} u^{\sigma+1} f^{(\sigma+1)}(u) = 0, \quad \sigma = r+1, r+2. \tag{2.11}$$

By [8, Lemma 2.2] (the lemma is applicable by virtue of (2.10) with $j = r + 2$), we have

$$\big(S_n f(x) - f(x)\big)^{(r)} = \frac{r}{2n} f^{(r+1)}(x) + \frac{1}{(r+1)!} \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{r+1} f^{(r+2)}(u)\, du, \quad x > 0.$$

Next, we integrate by parts the integrals twice, as for the term with $k = 0$ we take into consideration (2.10) with $j = r + 3$ and (2.11). Thus we arrive at

$$\begin{aligned}\big(S_n f(x) - f(x)\big)^{(r)} &= \frac{r}{2n} f^{(r+1)}(x) + \frac{1}{(r+2)!\, n^{r+2}} T_{r,n,r+2}(x) f^{(r+2)}(x)\\ &+ \frac{1}{(r+3)!\, n^{r+3}} T_{r,n,r+3}(x) f^{(r+3)}(x)\\ &+ \frac{1}{(r+3)!} \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_x^{k/n} \left(\frac{k}{n} - u\right)^{r+3} f^{(r+4)}(u)\, du, \quad x > 0.\end{aligned}$$

We will show that

$$\begin{aligned} T_{r,n,r+2}(x) &= n^r \left(r!\, S(r+2, r) + \frac{(r+2)!}{2}\, nx \right),\\ T_{r,n,r+3}(x) &= n^r \left(r!\, S(r+3, r) + \frac{(r+3)!\,(3r+2)}{12}\, nx \right). \end{aligned} \tag{2.12}$$

Then, since $(\widetilde{D} f)^{(r)}(x) = r f^{(r+1)}(x) + x f^{(r+2)}(x)$, we get the assertion of the lemma.

By virtue of [8, Lemma 2.1] with $\ell = r + 2, r + 3$, we have

$$T_{r,n,r+2}(x) = n^r (d_1 + d_2 nx)$$

and

$$T_{r,n,r+3}(x) = n^r (d_3 + d_4 nx),$$

where d_i, $i = 1, \dots, 4$ are constants whose value is independent of n (and x).

Clearly, $s_{n,k}^{(r)}(0) = (-1)^{r-k} n^r \binom{r}{k}$ for $0 \le k \le r$, and $s_{n,k}^{(r)}(0) = 0$ for $k > r$. Therefore,

$$d_1 = n^{-r} T_{r,n,r+2}(0) = \sum_{k=0}^{\infty} k^{r+2} s_{n,k}^{(r)}(0)$$

$$= \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^{r+2}$$
$$= r!\, S(r+2, r).$$

Just similarly, we get

$$d_3 = r!\, S(r+3, r).$$

To calculate d_2 we use analogous considerations and also $T_{r,n,r+1}(x) \equiv n^r (r+1)!\, r/2$ (see [8, Lemma 2.1]) to obtain

$$\begin{aligned} d_2 &= n^{-r-1} T'_{r,n,r+2}(x) \\ &= -n^{-r}(r+2) T_{r,n,r+1}(x) + n^{-r-1} T_{r+1,n,r+2}(x) \\ &= \frac{(r+2)!}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} d_4 &= n^{-r-1} T'_{r,n,r+3}(x) \\ &= -n^{-r}(r+3) T_{r,n,r+2}(x) + n^{-r-1} T_{r+1,n,r+3}(x) \\ &= r! \big[(r+1) S(r+3, r+1) - (r+3) S(r+2, r)\big] \\ &= \frac{(r+3)!\,(3r+2)}{12}. \end{aligned}$$

Above we have used that (see [11, Section 3.4])

$$\begin{aligned} S(r+2, r) &= \binom{r+2}{3} + 3\binom{r+2}{4} \\ &= \frac{r(r+1)(r+2)(3r+1)}{24}. \end{aligned} \tag{2.13}$$

This completes the proof of (2.12). □

Proposition 2.2. *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) with $0 \le \gamma_0 < r$ and $\gamma_\infty \in \mathbb{R}$. Then for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r+3}(0,\infty)$ and $wf^{(r+2)}, wf^{(r+3)}, w\varphi^4 f^{(r+4)} \in L_\infty[0,\infty)$ and all $n \ge 1$ there holds*

$$\begin{aligned} &\left\| w\left(S_n f - f - \frac{1}{2n}\widetilde{D} f\right)^{(r)} \right\| \\ &\qquad \le \frac{c}{n^2}\left(\|wf^{(r+2)}\| + \|w\varphi^2 f^{(r+3)}\| + \|w\varphi^4 f^{(r+4)}\|\right) + \frac{c}{n^3}\,\|wf^{(r+3)}\|. \end{aligned}$$

The constant $c > 0$ is independent of f and n.

Remark 2.3. Let us note that $wf^{(r+2)}, wf^{(r+3)}, w\varphi^4 f^{(r+4)} \in L_\infty[0,\infty)$ implies $w\varphi^2 f^{(r+3)} \in L_\infty[0,\infty)$. This can be shown by e.g. [9, Proposition 4.1] with $p=\infty$, $k=1$, r fixed to be equal to 2, $g=f^{(r+2)}$ and $a=1/2$ (or see [6, Lemma 1]), which yields

$$\|w\varphi^2 f^{(r+3)}\|_{[1/2,\infty)} \le c\left(\|wf^{(r+2)}\|_{[1/2,\infty)} + \|w\varphi^4 f^{(r+4)}\|_{[1/2,\infty)}\right). \tag{2.14}$$

Here $\|\circ\|_{[1/2,\infty)}$ stands for the essential supremum norm on the interval $[1/2,\infty)$.

Proof of Proposition 2.2. Note that $\varphi^{2r+6} f^{(r+4)} \in L[0,1]$. We set

$$\widetilde{R}_{r,n}(x) := \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x)\, \tilde{\rho}_{r,x}\left(\frac{k}{n}\right),$$

where

$$\tilde{\rho}_{r,x}(t) := \int_x^t (t-u)^{r+3} f^{(r+4)}(u)\,du. \tag{2.15}$$

In view of Lemma 2.1, we have

$$\left\| w\left(S_n f - f - \frac{1}{2n}\widetilde{D} f\right)^{(r)} \right\| \le \frac{c}{n^2}\left(\|wf^{(r+2)}\| + \|w\varphi^2 f^{(r+3)}\|\right) + \frac{c}{n^3}\|wf^{(r+3)}\| + \|w\widetilde{R}_{r,n}\|.$$

To complete the proof of the proposition, we will show that

$$\|w\widetilde{R}_{r,n}\| \le \frac{c}{n^3}\|wf^{(r+3)}\| + \frac{c}{n^2}\|w\varphi^4 f^{(r+4)}\|. \tag{2.16}$$

We use that

$$|\tilde{\rho}_{r,x}(t)| \le \left| \int_x^t \frac{|t-u|^{r+3}}{u^{\gamma_0+2}(1+u)^{\gamma_\infty-\gamma_0}}\,du \right| \|w\varphi^4 f^{(r+4)}\|. \tag{2.17}$$

By Hölder's inequality we arrive at

$$\left| \int_x^t \frac{|t-u|^{r+3}}{u^{\gamma_0+2}(1+u)^{\gamma_\infty-\gamma_0}}\,du \right| \le \left| \int_x^t \frac{|t-u|^{r+3}}{u^{p(\gamma_0+2)}}\,du \right|^{1/p} \left| \int_x^t \frac{|t-u|^{r+3}}{(1+u)^{q(\gamma_\infty-\gamma_0)}}\,du \right|^{1/q}, \tag{2.18}$$

where we have set $p := (r+3)/(\gamma_0+2)$ and q is its conjugate exponent.

It is quite straightforward to verify that

$$\frac{|t-u|}{u} \le \frac{|t-x|}{x}$$

for u between x and t. Therefore,

$$\left|\int_x^t \frac{|t-u|^{r+3}}{u^{p(\gamma_0+2)}}\,du\right|^{1/p} \le \frac{|t-x|^{(r+4)/p}}{x^{\gamma_0+2}}. \tag{2.19}$$

Clearly, if u is between x and t, then

$$(1+u)^\gamma \le (1+x)^\gamma + (1+t)^\gamma$$

for any $\gamma \in \mathbb{R}$. Consequently,

$$\left|\int_x^t \frac{|t-u|^{r+3}}{(1+u)^{q(\gamma_\infty-\gamma_0)}}\,du\right|^{1/q} \le \frac{|t-x|^{(r+4)/q}}{(1+x)^{\gamma_\infty-\gamma_0}} + \frac{|t-x|^{(r+4)/q}}{(1+t)^{\gamma_\infty-\gamma_0}}. \tag{2.20}$$

Combining (2.17)-(2.20), we arrive at the estimate

$$\begin{aligned}(2.21)\quad & |w(x)\tilde{\rho}_{r,x}(t)| \\ & \le \left(1 + \frac{(1+x)^{\gamma_\infty-\gamma_0}}{(1+t)^{\gamma_\infty-\gamma_0}}\right) \frac{|t-x|^{r+4}}{x^2}\,\|w\varphi^4 f^{(r+4)}\|, \quad x>0,\ t\ge 0.\end{aligned}$$

We consider two cases.

Case 1: $nx \ge 1$. Inequality (2.21) implies

$$\begin{aligned}(2.22)\quad |w(x)R_{r,n}(x)| &\le \frac{1}{x^2}\sum_{k=0}^\infty |s_{n,k}^{(r)}(x)| \left|\frac{k}{n}-x\right|^{r+4} \|w\varphi^4 f^{(r+4)}\| \\ &+ \frac{(1+x)^{\gamma_\infty-\gamma_0}}{x^2}\sum_{k=0}^\infty |s_{n,k}^{(r)}(x)| \left|\frac{k}{n}-x\right|^{r+4} \left(1+\frac{k}{n}\right)^{\gamma_0-\gamma_\infty} \|w\varphi^4 f^{(r+4)}\|.\end{aligned}$$

To estimate the first sum above, we apply (2.4) and (2.7) to deduce

$$\begin{aligned} & \frac{1}{x^2}\sum_{k=0}^\infty |s_{n,k}^{(r)}(x)| \left|\frac{k}{n}-x\right|^{r+4} \\ (2.23)\qquad & \le \frac{c}{n^2}\sum_{0\le i\le r/2} (nx)^{i-r-2} \sum_{j=0}^{r-2i}\sum_{k=0}^\infty |k-nx|^{r+j+4} s_{n,k}(x) \\ & \le \frac{c}{n^2}\sum_{0\le i\le r/2}\sum_{j=0}^{r-2i} (nx)^{(2i-r+j)/2} \le \frac{c}{n^2},\end{aligned}$$

where at the last inequality we have taken into consideration that $2i-r+j\le 0$ for all i and j in the specified range.

We estimate the other sum in (2.22) in a similar way, as we also use Cauchy's inequality on the sum on k in order to split $|k-nx|^{r+j+4}$ and $(1+k/n)^{\gamma_0-\gamma_\infty}$. We have

(2.24)
$$\begin{aligned}
&\frac{(1+x)^{\gamma_\infty-\gamma_0}}{x^2}\sum_{k=0}^{\infty}|s_{n,k}^{(r)}(x)|\left|\frac{k}{n}-x\right|^{r+4}\left(1+\frac{k}{n}\right)^{\gamma_0-\gamma_\infty}\\
&\le \frac{c\,(1+x)^{\gamma_\infty-\gamma_0}}{n^2}\sum_{0\le i\le r/2}(nx)^{i-r-2}\sum_{j=0}^{r-2i}\sum_{k=0}^{\infty}|k-nx|^{r+j+4}\left(1+\frac{k}{n}\right)^{\gamma_0-\gamma_\infty}s_{n,k}(x)\\
&\le \frac{c\,(1+x)^{\gamma_\infty-\gamma_0}}{n^2}\sum_{0\le i\le r/2}(nx)^{i-r-2}\sum_{j=0}^{r-2i}\sqrt{\sum_{k=0}^{\infty}|k-nx|^{2(r+j+4)}s_{n,k}(x)}\\
&\qquad\times\sqrt{\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{2(\gamma_0-\gamma_\infty)}s_{n,k}(x)}.
\end{aligned}$$

By (2.7), we have

$$\sum_{k=0}^{\infty}|k-nx|^{2(r+j+4)}s_{n,k}(x)\le c\,(nx)^{r+j+4},\quad nx\ge 1. \tag{2.25}$$

It was shown in [5, p. 163] that

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{m}s_{n,k}(x)\le c\,(1+x)^m,\quad x\ge 0,\quad m\in\mathbb{Z}. \tag{2.26}$$

Then by means of Hölder's inequality and the identity $\sum_{k=0}^{\infty}s_{n,k}(x)\equiv 1$ we derive (see [5, p. 162–163])

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{2(\gamma_0-\gamma_\infty)}s_{n,k}(x)\le c\,(1+x)^{2(\gamma_0-\gamma_\infty)},\quad x\ge 0. \tag{2.27}$$

Combining (2.24), (2.25) and (2.27), we arrive at

$$\frac{(1+x)^{\gamma_\infty-\gamma_0}}{x^2}\sum_{k=0}^{\infty}|s_{n,k}^{(r)}(x)|\left|\frac{k}{n}-x\right|^{r+4}\left(1+\frac{k}{n}\right)^{\gamma_0-\gamma_\infty}\le\frac{c}{n^2}.$$

Now, (2.22), (2.23) and the last estimate above yield

$$|w(x)\widetilde{R}_{r,n}(x)| \le \frac{c}{n^2}\,\|w\varphi^4 f^{(r+4)}\|, \quad nx \ge 1. \tag{2.28}$$

Case 2: $nx \le 1$. By means of (2.3) and summation by parts we derive for $n \ge 1$ the relation (cf. (2.8))

$$\widetilde{R}_{r,n}(x) = n^r \sum_{k=0}^{\infty} \overrightarrow{\Delta}^r_{1/n}\tilde{\rho}_{r,x}\left(\frac{k}{n}\right) s_{n,k}(x).$$

Consequently,

$$|w(x)\widetilde{R}_{r,n}(x)| \le c\,n^r \max_{i=0,\dots,r} \sum_{k=0}^{\infty} \left| w(x)\,\tilde{\rho}_{r,x}\left(\frac{k+i}{n}\right)\right| s_{n,k}(x). \tag{2.29}$$

We will estimate the terms for $k=0$ and $k=1$ separately. For the sum on $k \ge 2$, we apply (2.21) and Cauchy's inequality to arrive at

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left| w(x)\,\tilde{\rho}_{r,x}\left(\frac{k+i}{n}\right)\right| s_{n,k}(x) \\
&\quad \le \frac{1}{x^2}\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{r+4} s_{n,k}(x)\,\|w\varphi^4 f^{(r+4)}\| \\
&\qquad + \frac{(1+x)^{\gamma_\infty-\gamma_0}}{x^2}\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{r+4}\left(1+\frac{k+i}{n}\right)^{\gamma_0-\gamma_\infty} s_{n,k}(x)\,\|w\varphi^4 f^{(r+4)}\| \\
&\quad \le \frac{1}{x^2}\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{r+4} s_{n,k}(x)\,\|w\varphi^4 f^{(r+4)}\| \\
&\qquad + \frac{c}{x^2}\sqrt{\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{2(r+4)} s_{n,k}(x)} \\
&\qquad\quad \times \sqrt{\sum_{k=2}^{\infty}\left(1+\frac{k+i}{n}\right)^{2(\gamma_0-\gamma_\infty)} s_{n,k}(x)}\,\|w\varphi^4 f^{(r+4)}\|.
\end{aligned}$$

We will show that

$$\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{l} s_{n,k}(x) \le \frac{c\,x^2}{n^{l-2}}, \quad l \in \mathbb{N}_+,\ l \ge 2, \tag{2.30}$$

and

$$\sum_{k=2}^{\infty}\left(1+\frac{k+i}{n}\right)^{\gamma} s_{n,k}(x) \le c\,(nx)^2, \quad \gamma\in\mathbb{R}, \tag{2.31}$$

for $nx \le 1$ and $i = 0, \dots, r$.

Then we will get

$$\sum_{k=2}^{\infty}\left|w(x)\,\tilde{\rho}_{r,x}\left(\frac{k+i}{n}\right)\right| s_{n,k}(x) \le \frac{c}{n^{r+2}}\,\|w\varphi^4 f^{(r+4)}\|, \quad i=0,\dots,r. \tag{2.32}$$

To verify (2.30)–(2.31), we apply [8, (3.16) and (3.17)] to the right-hand side of the trivial inequalities

$$\sum_{k=2}^{\infty}\left(\frac{k+i}{n}-x\right)^{l} s_{n,k}(x) \le nx\sum_{k=1}^{\infty}\left(\frac{k+i}{n}-x\right)^{l} s_{n,k}(x)$$

and

$$\sum_{k=2}^{\infty}\left(1+\frac{k+i}{n}\right)^{\gamma} s_{n,k}(x) \le nx\sum_{k=1}^{\infty}\left(1+\frac{k+i}{n}\right)^{\gamma} s_{n,k}(x),$$

where $0 \le x \le 1/n$, $l \in \mathbb{N}_+$ and $\gamma \in \mathbb{R}$.

Now, let us consider the terms for $k = 0, 1$ in (2.29). For $k = 0$ and $i = 0$ we again use (2.21) to get directly

$$\begin{aligned} |w(x)\,\tilde{\rho}_{r,x}(0)| &\le c\,x^{r+2}\|w\varphi^4 f^{(r+4)}\| \\ &\le \frac{c}{n^{r+2}}\,\|w\varphi^4 f^{(r+4)}\|. \end{aligned} \tag{2.33}$$

It remains to estimate $\tilde{\rho}_{r,x}(i/n)$, defined in (2.15), for $i = 1, \dots, r+1$. To this end, we expand $(i/n - u)^{r+3}$ by the binomial formula to get

$$\left|w(x)\tilde{\rho}_{r,x}\left(\frac{i}{n}\right)\right| \le c\,x^{\gamma_0}\sum_{j=0}^{r+3}\frac{1}{n^{r-j+3}}\left|\int_x^{i/n} u^j f^{(r+4)}(u)\,du\right|. \tag{2.34}$$

Clearly, for $j = 2, \dots, r+3$ we have

$$\begin{aligned} x^{\gamma_0}\left|\int_x^{i/n} u^j f^{(r+4)}(u)\,du\right| &\le c\,x^{\gamma_0}\int_x^{i/n} u^{j-\gamma_0-2}du\,\|w\varphi^4 f^{(r+4)}\| \\ &\le \frac{c\,x^{\gamma_0}}{n}\left(\frac{1}{n^{j-\gamma_0-2}}+x^{j-\gamma_0-2}\right)\|w\varphi^4 f^{(r+4)}\| \end{aligned}$$

$$\leq \frac{c}{n^{j-1}} \|w\varphi^4 f^{(r+4)}\|, \quad x \in (0, 1/n].$$

For the integral in (2.34) with $j = 0$ we have

$$\begin{aligned}
x^{\gamma_0} \left| \int_x^{i/n} f^{(r+4)}(u)\, du \right| &= x^{\gamma_0} \left| f^{(r+3)}\left(\frac{i}{n}\right) - f^{(r+3)}(x) \right| \\
&\leq \left(\frac{i}{n}\right)^{\gamma_0} \left| f^{(r+3)}\left(\frac{i}{n}\right) \right| + x^{\gamma_0} |f^{(r+3)}(x)| \\
&\leq c\, \|wf^{(r+3)}\|, \quad x \in (0, 1/n].
\end{aligned}$$

Similarly, for the integral with $j = 1$, we have, after integrating by parts,

$$\begin{aligned}
&x^{\gamma_0} \left| \int_x^{i/n} u f^{(r+4)}(u)\, du \right| = x^{\gamma_0} \left| \int_x^{i/n} u\, d\, f^{(r+3)}(u) \right| \\
&\leq \frac{1}{n} \left[\left(\frac{i}{n}\right)^{\gamma_0} \left| f^{(r+3)}\left(\frac{i}{n}\right) \right| + x^{\gamma_0} |f^{(r+3)}(x)| \right] + x^{\gamma_0} \int_x^{i/n} |f^{(r+3)}(u)|\, du \\
&\leq \frac{c}{n} \|wf^{(r+3)}\|, \quad x \in (0, 1/n].
\end{aligned}$$

Thus we have established for $nx \leq 1$ and $i = 1, \dots, r+1$

$$\left| w(x) \tilde{\rho}_{r,x}\left(\frac{i}{n}\right) \right| \leq \frac{c}{n^{r+3}} \|wf^{(r+3)}\| + \frac{c}{n^{r+2}} \|w\varphi^4 f^{(r+4)}\|. \tag{2.35}$$

Inequalities (2.29), (2.32), (2.33) and (2.35) yield

$$|w(x) \widetilde{R}_{r,n}(x)| \leq \frac{c}{n^3} \|wf^{(r+3)}\| + \frac{c}{n^2} \|w\varphi^4 f^{(r+4)}\|, \quad nx \leq 1.$$

This along with (2.28) completes the proof of (2.16). □

Similar point-wise Voronovskaya-type estimates were established in [1, Theorem 2] for any $r \in \mathbb{N}_0$ and $w(x) := (1+x)^{-2}$, and also in [2] for general linear positive operators, which in particular include S_n, for the first and second derivative and weights $w(x) := (1+x)^{-m}$, where $m \in \mathbb{N}_+$.

We proceed to several Bernstein-type inequalities.

Proposition 2.4. *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \leq \gamma_0 < r$ and $\gamma_\infty \in \mathbb{R}$. Then for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \geq 1$ there hold:*

(a) $\|w(S_n f)^{(r+1)}\| \leq cn \|wf^{(r)}\|$;

(b) $\|w\varphi^2(S_nf)^{(r+2)}\| \le cn\|wf^{(r)}\|$.

Proof. (a) By virtue of (2.8) with $r+1$ in place of r, we have

$$\begin{aligned}|(S_nf)^{(r+1)}(x)| &= n^{r+1}\left|\sum_{k=0}^{\infty} \overrightarrow{\Delta}_{1/n}^{r+1} f\left(\frac{k}{n}\right) s_{n,k}(x)\right| \\ &\le 2n^{r+1}\max_{j=0,1}\sum_{k=0}^{\infty}\left|\overrightarrow{\Delta}_{1/n}^{r} f\left(\frac{k+j}{n}\right)\right| s_{n,k}(x), \quad x\ge 0.\end{aligned}$$

Let us recall that (see e.g. [3, p. 45])

$$\overrightarrow{\Delta}_{h}^{r} f(x) = h^r\int_0^r M_r(u)f^{(r)}(x+hu)\,du, \quad x\ge 0,$$

where M_r is the r-fold convolution of the characteristic function of $[0,1]$ with itself and

$$0\le M_r(u)\le c\,u^{r-1}, \quad u\in[0,r].$$

Therefore,

$$\left|\overrightarrow{\Delta}_{1/n}^{r} f\left(\frac{k}{n}\right)\right| \le \frac{c}{n^r}\int_0^r \frac{u^{r-1}}{w\left(\frac{k+u}{n}\right)}\,du\,\|wf^{(r)}\|, \quad k\in\mathbb{N}_0. \tag{2.36}$$

Consequently,

$$\begin{aligned}&|w(x)(S_nf)^{(r+1)}(x)| \\ &\qquad \le cnw(x)\max_{j=0,1}\sum_{k=0}^{\infty}\int_0^r \frac{u^{r-1}}{w\left(\frac{k+j+u}{n}\right)}\,du\,s_{n,k}(x)\,\|wf^{(r)}\|, \quad x\ge 0.\end{aligned} \tag{2.37}$$

It is quite straightforward to obtain (see [8, Proposition 3.1]) that

$$\int_0^r \frac{u^{r-1}}{w\left(\frac{k+u}{n}\right)}\,du \le c\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}, \quad k\ge 0; \tag{2.38}$$

hence,

$$\int_0^r \frac{u^{r-1}}{w\left(\frac{k+u+1}{n}\right)}\,du \le c\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}, \quad k\ge 0, \tag{2.39}$$

as well.

It was shown in [5, (10.2.4)] that

$$\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{l} s_{n,k}(x) \le c\,x^{-l}, \quad x>0, \quad l\in\mathbb{N}_0,$$

This along with (2.26), the identity $\sum_{k=0}^{\infty} s_{n,k}(x)\equiv 1$ and Hölder's inequality yields (see [5, p. 162–163])

$$\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0} s_{n,k}(x) \le \frac{c}{w(x)}, \quad x>0, \tag{2.40}$$

for all $\gamma_0\ge 0$ and $\gamma_\infty\in\mathbb{R}$.

Estimates (2.37)-(2.40) imply (a).

(b) As in the proof of Proposition 2.2 we consider the cases $nx\ge 1$ and $nx\le 1$ separately.

Case 1: $nx\ge 1$. We differentiate identity (2.8) twice to get

$$(S_n f)^{(r+2)}(x) = n^r \sum_{k=0}^{\infty} \overrightarrow{\Delta}_{1/n}^{r} f\left(\frac{k}{n}\right) s''_{n,k}(x).$$

We note that the series on the right-hand side of (2.8) can be differentiated term-by-term any number of times because, under the assumptions on f, the resulting series are uniformly convergent on any finite closed subinterval of $[0,\infty)$, as can be shown by means of the Weierstrass M-test.

Using (2.2) (cf. (2.4) with $r=2$), we compute that

$$s''_{n,k}(x) = \frac{s_{n,k}(x)}{x^2}\left(-(k-nx)+(k-nx)^2-nx\right), \quad k\in\mathbb{N}_0.$$

Therefore,

$$\begin{aligned}&|w(x)\varphi^2(x)(S_n f)^{(r+2)}(x)|\\ &\quad\le n^r\frac{w(x)}{x}\sum_{k=0}^{\infty}\left|\overrightarrow{\Delta}_{1/n}^{r} f\left(\frac{k}{n}\right)\right|\left(|k-nx|+(k-nx)^2+nx\right) s_{n,k}(x), \quad x>0.\end{aligned}$$

Then we combine (2.36) and (2.38) to estimate $|\overrightarrow{\Delta}_{1/n}^{r} f(k/n)|$ and derive the inequality

(2.41)
$$\begin{aligned}|w(x)\varphi^2(x)(S_nf)^{(r+2)}(x)| &\le c\frac{w(x)}{x}\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}|k-nx|s_{n,k}(x)\,\|wf^{(r)}\|\\ &+c\frac{w(x)}{x}\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}(k-nx)^2 s_{n,k}(x)\,\|wf^{(r)}\|\\ &+cnw(x)\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}s_{n,k}(x)\,\|wf^{(r)}\|.\end{aligned}$$

We further estimate the first two sums above, using Cauchy's inequality (2.40) with $2\gamma_0$ in place of γ_0 and $2\gamma_\infty$ in place of γ_∞, and (2.6), to arrive at

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}|k-nx|s_{n,k}(x)\\ &\le\sqrt{\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{2\gamma_0}\left(\frac{n}{n+k}\right)^{2(\gamma_\infty-\gamma_0)}s_{n,k}(x)}\sqrt{T_{n,2}(x)}\\ &\le c\sqrt{w^{-2}(x)}\sqrt{nx}\le c\frac{nx}{w(x)}\end{aligned}\tag{2.42}$$

and

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{\gamma_0}\left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0}(k-nx)^2 s_{n,k}(x)\\ &\le\sqrt{\sum_{k=0}^{\infty}\left(\frac{n}{k+1}\right)^{2\gamma_0}\left(\frac{n}{n+k}\right)^{2(\gamma_\infty-\gamma_0)}s_{n,k}(x)}\sqrt{T_{n,4}(x)}\\ &\le c\sqrt{w^{-2}(x)}nx=c\frac{nx}{w(x)}.\end{aligned}\tag{2.43}$$

Now, combining (2.41) with (2.42), (2.43) and (2.40), we get

$$|w(x)\varphi^2(x)(S_nf)^{(r+2)}(x)|\le cn\,\|wf^{(r)}\|,\quad nx\ge 1.\tag{2.44}$$

Case 2: $nx\le 1$. We differentiate identity (2.8) with $r+1$ in place of r

and thus get

$$(S_n f)^{(r+2)}(x) = n^{r+1} \sum_{k=0}^{\infty} \overrightarrow{\Delta}_{1/n}^{r+1} f\left(\frac{k}{n}\right) s'_{n,k}(x).$$

Then we use (2.2), (2.36), (2.38), (2.39) and (2.42) to get

$$\begin{aligned}
&|w(x)\varphi^2(x)(S_n f)^{(r+2)}(x)| \\
&\quad \le 2n^{r+1} w(x) \max_{j=0,1} \sum_{k=0}^{\infty} \left| \overrightarrow{\Delta}_{1/n}^{r} f\left(\frac{k+j}{n}\right)\right| |k-nx| s_{n,k}(x) \\
&\quad \le cnw(x) \sum_{k=0}^{\infty} \left(\frac{n}{k+1}\right)^{\gamma_0} \left(\frac{n}{n+k}\right)^{\gamma_\infty-\gamma_0} |k-nx| s_{n,k}(x)\, \|wf^{(r)}\| \\
&\quad \le cnw(x) \frac{nx}{w(x)}\, \|wf^{(r)}\| \\
&\quad \le cn\, \|wf^{(r)}\|, \quad x \in (0, 1/n].
\end{aligned}$$

where at the last estimate we have taken into consideration that $nx \le 1$.

Thus we have established

$$|w(x)\varphi^2(x)(S_n f)^{(r+2)}(x)| \le cn\, \|wf^{(r)}\|, \quad nx \le 1. \tag{2.45}$$

Estimates (2.44) and (2.45) verify assertion (b). □

Since $\big(\widetilde{D}g\big)^{(r)} = rg^{(r+1)} + \varphi^2 g^{(r+2)}$, Proposition 2.4 immediately yields the following inequality.

Corollary 2.5. *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \in \mathbb{R}$. Then for all $f \in C[0,\infty)$ such that $f \in AC_{loc}^{r-1}(0,\infty)$ and $wf^{(r)} \in L_\infty[0,\infty)$, and all $n \ge 1$ there holds*

$$\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\| \le cn\|wf^{(r)}\|.$$

We will also use the following inequalities, which follow from Proposition 2.4 and the embedding inequalities [8, Proposition 2.4].

Corollary 2.6. *Let $r \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_\infty)$ be given by (1.1) as $0 \le \gamma_0 < r$ and $\gamma_\infty \ne r$. Then for all $f \in AC^{r+1}[0,\infty)$ such that $wf^{(r)} \in L_\infty[0,\infty)$ and $w\big(\widetilde{D}f\big)^{(r)} \in L_\infty[0,\infty)$, and all $n \ge 1$ there hold:*

(a) $\|w(S_n f)^{(r+2)}\| \le cn\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|$;

(b) $\|w(S_n^2 f)^{(r+3)}\| \le cn^2\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|$;

(c) $\|w\varphi^2(S_nf)^{(r+3)}\| \le cn\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|$;

(d) $\|w\varphi^4(S_nf)^{(r+4)}\| \le cn\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|$.

Proof. (a) By virtue of [8, (2.15)], we have

$$\|wf^{(r+1)}\| \le c\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|. \tag{2.46}$$

This shows, in the first place, that $wf^{(r+1)} \in L_\infty[0,\infty)$. Then we apply Proposition 2.4(a) with $r+1$ in place of r to get

$$\|w(S_nf)^{(r+2)}\| \le cn\|wf^{(r+1)}\|,$$

which combined with (2.46) yields (a).

(b) The assertion follows from Proposition 2.4(a) with $r+2$ in place of r and S_nf in place of f and (a).

(c) Similarly to (a), we apply Proposition 2.4(b) with $r+1$ in place of r and (2.46) to derive

$$\begin{aligned}\|w\varphi^2(S_nf)^{(r+3)}\| &\le cn\|wf^{(r+1)}\|\\ &\le cn\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|.\end{aligned}$$

(d) We apply Proposition 2.4(b) with $r+2$ in place of r and $w\varphi^2$ in place of w. Thus we get

$$\|w\varphi^4(S_nf)^{(r+4)}\| \le cn\|w\varphi^2f^{(r+2)}\|. \tag{2.47}$$

Let us note that the assumption in Proposition 2.4(b) on the weight exponent at 0 now is $0 \le \gamma_0 + 1 < r+2$, which is satisfied. As for the assumptions on the function, it remains only to observe that $w\varphi^2f^{(r+2)} \in L_\infty[0,\infty)$. It follows from [8, (2.16)], by virtue of which we have

$$\|w\varphi^2f^{(r+2)}\| \le c\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|.$$

The last estimate and (2.47) yield (d). □

3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We apply the method to establish converse inequalities given in [4, Theorem 3.2]. This theorem is not directly applicable because the Voronovskaya-type estimate has a different form—compare [4, (3.4)] and Proposition 2.2. However, the same idea still works.

We set $g_n := S_n^3 f$. First, we will show that g_n is in the domain on which the infimum in the definition of the K-functional $\widetilde{K}_r(f^{(r)},t)_w$ is taken and hence

$$\widetilde{K}_r(f^{(r)},n^{-1})_w \le \|w(f^{(r)}-g_n^{(r)})\| + \frac{1}{n}\big\|w\big(\widetilde{D}g_n\big)^{(r)}\big\|. \tag{3.1}$$

Indeed, clearly, $g_n \in AC^{r+1}[0,\infty)$. Next, iterating (2.9), we see that $wg_n^{(r)} \in L_\infty[0,\infty)$, whereas $w\big(\widetilde{D}g_n\big)^{(r)} \in L_\infty[0,\infty)$ follows from Corollary 2.5 and (2.9), which imply

$$\begin{aligned}\big\|w\big(\widetilde{D}g_n\big)^{(r)}\big\| &= \big\|w\big(\widetilde{D}S_n^3 f\big)^{(r)}\big\|\\ &\le cn\|wS_n^2 f^{(r)}\|\\ &\le cn\|wf^{(r)}\|.\end{aligned}$$

Let I stand for the identity map in the L_∞-space with the weight w on $[0,\infty)$. We have, by virtue of (2.9),

$$\begin{aligned}\|w(f^{(r)}-g_n^{(r)})\| &= \big\|w\big[(I+S_n+S_n^2)(f-S_nf)\big]^{(r)}\big\|\\ &\le c\,\|w(f-S_nf)^{(r)}\|.\end{aligned} \tag{3.2}$$

To complete the proof of the theorem, we will show that there exists $R \ge 1$ such that for all $n,k \ge 1$ such that $k \ge Rn$ there holds

$$\frac{1}{n}\big\|w\big(\widetilde{D}g_n\big)^{(r)}\big\| \le c\frac{k}{n}\Big(\|w(S_nf-f)^{(r)}\| + \|w(S_kf-f)^{(r)}\|\Big). \tag{3.3}$$

Then the first assertion of Theorem 1.1 follows from (3.1)-(3.3).

Let $k \ge n \ge 1$. We want to apply Proposition 2.2 with g_n in place of f. To this end, we first verify that $wg_n^{(r+2)}, wg_n^{(r+3)}, w\varphi^4 g_n^{(r+4)} \in L_\infty[0,\infty)$. To show it and, moreover, get estimates of their weighted L_∞-norms, we apply Corollary 2.6, (a), (b) and (d) with S_nf in place of f (note that $w\big(\widetilde{D}S_nf\big)^{(r)} \in L_\infty[0,\infty)$ by Corollary 2.5). Thus we get

$$\|w(S_n^2 f)^{(r+2)}\| \le cn\big\|w\big(\widetilde{D}S_nf\big)^{(r)}\big\|, \tag{3.4}$$

$$\|w(S_n^3 f)^{(r+3)}\| \le cn^2\big\|w\big(\widetilde{D}S_nf\big)^{(r)}\big\|, \tag{3.5}$$

and

$$\|w\varphi^4(S_n^2 f)^{(r+4)}\| \le cn\big\|w\big(\widetilde{D}S_nf\big)^{(r)}\big\|. \tag{3.6}$$

Further, by means of (2.9) with $S_n^2 f$ in place of f, we get from (3.4) and (3.6)

$$\|w(S_n^3 f)^{(r+2)}\| \le c\,\|w(S_n^2 f)^{(r+2)}\| \le cn\big\|w\big(\widetilde{D}S_nf\big)^{(r)}\big\|, \tag{3.7}$$

and

$$\|w\varphi^4(S_n^3 f)^{(r+4)}\| \le c\,\|w\varphi^4(S_n^2 f)^{(r+4)}\| \le cn\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\|. \tag{3.8}$$

For the application of (2.9) in the latter case, we observe that the assumption on the weight exponent at 0 is $0 \le \gamma_0 + 2 < r + 4$, which is satisfied.

Having verified that $wg_n^{(r+2)}, wg_n^{(r+3)}, w\varphi^4 g_n^{(r+4)} \in L_\infty[0,\infty)$, we next apply Proposition 2.2 with k in place of n and g_n in place of f to arrive at

(3.9)

$$\begin{aligned}\frac{1}{n}\big\|w\big(\widetilde{D}g_n\big)^{(r)}\big\| &\le \frac{2k}{n}\left\|w\left(S_k(S_n^3 f) - S_n^3 f - \frac{1}{2k}\,\widetilde{D}(S_n^3 f)\right)^{(r)}\right\| \\ &\quad + \frac{2k}{n}\left\|w\left(S_k(S_n^3 f) - S_n^3 f\right)^{(r)}\right\| \\ &\le \frac{c}{nk}\left(\|w(S_n^3 f)^{(r+2)}\| + \|w\varphi^2(S_n^3 f)^{(r+3)}\| + \|w\varphi^4(S_n^3 f)^{(r+4)}\|\right) \\ &\quad + \frac{c}{nk^2}\,\|w(S_n^3 f)^{(r+3)}\| + \frac{2k}{n}\left\|w\left(S_k(S_n^3 f) - S_n^3 f\right)^{(r)}\right\|.\end{aligned}$$

We will estimate the terms on the right.

Similarly as above, we use (2.9) with $w\varphi^2$ in place of w and $S_n^2 f$ in place of f, and Corollary 2.6(c) with $S_n f$ in place of f to get

$$\begin{aligned}\|w\varphi^2(S_n^3 f)^{(r+3)}\| &\le c\,\|w\varphi^2(S_n^2 f)^{(r+3)}\| \\ &\le cn\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\|.\end{aligned} \tag{3.10}$$

Here the application of (2.9) is justified since the assumption on the weight exponent at 0 is $0 \le \gamma_0 + 1 < r + 3$, which is clearly satisfied.

By virtue of (3.7), (3.10) and (3.8), we have

$$\begin{aligned}\frac{1}{nk}&\left(\|w(S_n^3 f)^{(r+2)}\| + \|w\varphi^2(S_n^3 f)^{(r+3)}\| + \|w\varphi^4(S_n^3 f)^{(r+4)}\|\right) \\ &\le \frac{c}{k}\,\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\|.\end{aligned} \tag{3.11}$$

Also, by (3.5), we get

$$\frac{1}{nk^2}\,\|w(S_n^3 f)^{(r+3)}\| \le \frac{c}{k}\,\big\|w\big(\widetilde{D}S_n f\big)^{(r)}\big\|, \tag{3.12}$$

where we have also taken into account that $n \le k$.

To estimate the last term on the right of (3.9) we use the representation

$$S_k(S_n^3 f) - S_n^3 f = S_k(S_n^3 f - f) + (S_k f - f) + (f - S_n^3 f).$$

Therefore, using also (2.9) and (3.2), we arrive at

$$\left\| w\left(S_k(S_n^3 f) - S_n^3 f\right)^{(r)} \right\| \le c\left(\|w(S_n f - f)^{(r)}\| + \|w(S_k f - f)^{(r)}\| \right). \tag{3.13}$$

We combine (3.9) with (3.11)-(3.13) to derive

$$\begin{aligned} &\frac{1}{n}\|w(\widetilde{D}g_n)^{(r)}\| \\ &\qquad \le \frac{c}{k}\left\|w(\widetilde{D}S_n f)^{(r)}\right\| + c\frac{k}{n}\left(\|w(S_n f - f)^{(r)}\| + \|w(S_k f - f)^{(r)}\| \right). \end{aligned} \tag{3.14}$$

Next, we will relate $\|w(\widetilde{D}S_n f)^{(r)}\|$ to $\|w(\widetilde{D}g_n)^{(r)}\|$. Using Corollary 2.5 and (2.9), we get

$$\begin{aligned} \|w(\widetilde{D}S_n f)^{(r)}\| &\le \|w(\widetilde{D}S_n^3 f)^{(r)}\| + \|w[\widetilde{D}S_n(f - S_n^2 f)]^{(r)}\| \\ &\le \|w(\widetilde{D}g_n)^{(r)}\|| + cn\,\|w(f - S_n^2 f)^{(r)}\| \\ &\le \|w(\widetilde{D}g_n)^{(r)}\| + cn\,\|w[(I + S_n)(f - S_n f)]^{(r)}\| \\ &\le \|w(\widetilde{D}g_n)^{(r)}\| + cn\,\|w(S_n f - f)^{(r)}\|. \end{aligned}$$

Hence (3.14) yields

$$\begin{aligned} &\frac{1}{n}\|w(\widetilde{D}g_n)^{(r)}\| \\ &\qquad \le \frac{c}{k}\left\|w(\widetilde{D}g_n)^{(r)}\right\| + c\frac{k}{n}\left(\|w(S_n f - f)^{(r)}\| + \|w(S_k f - f)^{(r)}\| \right) \end{aligned} \tag{3.15}$$

for all $k \ge n \ge 1$.

Let $R \ge 1$ and $k \ge Rn$. Then

$$\frac{c}{k} \le \frac{c}{Rn},$$

where c is the constant in (3.15). We fix R so large that $c/R \le 1/2$. Then (3.15) implies

$$\begin{aligned} &\frac{1}{n}\|w(\widetilde{D}g_n)^{(r)}\| \\ &\qquad \le \frac{1}{2n}\left\|w(\widetilde{D}S_n f)^{(r)}\right\| + c\frac{k}{n}\left(\|w(S_n f - f)^{(r)}\| + \|w(S_k f - f)^{(r)}\| \right) \end{aligned}$$

for all $n, k \ge 1$ such that $k \ge Rn$; hence the first assertion of the theorem follows. □

In the proof of Theorem 1.2 we will make use of the K-functionals

$$K_{2,\varphi}(f,t)_w := \inf\left\{ \|w(f - g)\| + t\|w\varphi^2 g''\| \right.$$

$$: g \in AC^1_{loc}(0,\infty),\ wg, w\varphi^2 g'' \in L_\infty[0,\infty)\big\}$$

and

$$K_1(f,t)_w := \inf\big\{\|w(f-g)\| + t\|wg'\| \\ : g \in AC_{loc}(0,\infty),\ wg, wg' \in L_\infty[0,\infty)\big\},$$

where $wf \in L_\infty[0,\infty)$ and $t > 0$.

Ditzian and Totik [5, Theorem 6.1.1] showed that there exist positive constants c and t_0 such that for all f with $wf \in L_\infty[0,\infty)$ and all $t \in (0, t_0]$ there holds

$$c^{-1}\omega^2_\varphi(f,t)_w \le K_{2,\varphi}(f,t^2)_w \le c\,\omega^2_\varphi(f,t)_w. \tag{3.16}$$

Analogously to the unweighted case (see e.g. [3, Chapter 6, Theorem 2.4]), we have

$$c^{-1}\omega(f,t)_w \le K_1(f,t)_w \le c\,\omega(f,t)_w, \quad t > 0. \tag{3.17}$$

P r o o f o f T h e o r e m 1.2. In view of Theorem 1.1 and the left inequalities in (3.16)–(3.17), it is sufficient to show that

$$K_{2,\varphi}(f,t)_w \le c\,\widetilde{K}_r(f,t)_w \tag{3.18}$$

and

$$K_1(f,t)_w \le c\,\widetilde{K}_r(f,t)_w, \tag{3.19}$$

where $wf \in L_\infty[0,\infty)$ and $t > 0$.

Let $g \in AC^{r+1}[0,\infty)$ with $wg^{(r)}, w\big(\widetilde{D}g\big)^{(r)} \in L_\infty[0,\infty)$ be arbitrarily fixed. Then, clearly, $g^{(r)} \in AC^1_{loc}(0,\infty)$. By virtue of [8, (2.16)], we have

$$\|w\varphi^2 g^{(r+2)}\| \le c\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|.$$

This implies that $w\varphi^2\big(g^{(r)}\big)'' \in L_\infty[0,\infty)$ and

$$K_{2,\varphi}(f,t)_w \le \|f - g^{(r)}\| + t\,\|w\varphi^2\big(g^{(r)}\big)''\| \\ \le c\left(\|f - g^{(r)}\| + t\big\|w\big(\widetilde{D}f\big)^{(r)}\big\|\right).$$

Taking the infimum on g, we straightforwardly arrive at (3.18).

Relation (3.19) is established just similarly by means of [8, (2.15)]. □

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Department of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd
1164 Sofia, Bulgaria
and
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: `bdraganov@fmi.uni-sofia.bg`

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