The trigonometric analogue of Taylor's formula and its application

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Abstract. A new approach to establishing generalized Taylor's expansions is used to prove the trigonometric analogue of Taylor's formula. We derive point-wise estimates of the error in the trigonometric interpolation and approximation by convolutional linear operators.

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1 Introduction

We consider the function spaces $L_p^*[-\pi,\pi]$, $1 \le p < \infty$, and $C^*[-\pi,\pi]$, where

$$\begin{aligned} L_p^*[-\pi,\pi] &= \{f : \mathbf{R} \to \mathbf{R} : f(x+2\pi) = f(x) \text{ a.e., } f|_{[-\pi,\pi]} \in L_p[-\pi,\pi]\}, \\ C^*[-\pi,\pi] &= \{f \in C(\mathbf{R}) : f(x+2\pi) = f(x)\}, \end{aligned}$$

normed, respectively, with the usual L_p-norm over the interval $[-\pi,\pi]$ for $1 \le p < \infty$, denoted by $\|\cdot\|_p$, and the uniform norm over the interval $[-\pi,\pi]$, denoted by $\|\cdot\|_\infty$.

In a recent paper (see [1]) we have introduced a new modulus of smoothness, which describes the rate of the best trigonometric approximation. It is defined by

$$\omega_r^T(f;t)_p := \sup_{0<h\le t} \|\Delta_h^{2r-1}\mathcal{F}_{r-1}f\|_p, \ r = 1,2,\dots,$$

where

$$\Delta_h^{2r-1}f(x) := \sum_{k=0}^{2r-1}(-1)^k\binom{2r-1}{k}f(x+((2r-1)/2-k)h)$$

is the symmetric finite difference of order $2r-1$,

$$\mathcal{F}_{r-1}(f,x) = f(x) + \int_0^x \mathcal{K}_{r-1}(t)f(x-t)\,dt$$

and

$$\mathcal{K}_{r-1}(t) = \sum_{j=1}^{r-1}\frac{a_j^{(r-1)}}{(2j-1)!}t^{2j-1}, \quad a_j^{(r-1)} = \sum_{1\le l_1<\dots<l_j\le r-1}(l_1\cdots l_j)^2.$$

It is shown in [1] that for the rate of the best trigonometric approximation $E_n^T(f)_p := \inf_{\tau\in T_n}\|f-\tau\|_p$, T_n being the set of all trigonometric polynomials of degree at most n, we have

$$E_n^T(f)_p \le C_r\omega_r^T(f;n^{-1})_p, \quad n\ge r-1, \tag{1.1}$$

and

$$\omega_r^T(f;t)_p \le C_r t^{2r-1}\sum_{r-1\le k\le 1/t}(k+1)^{2r-2}E_k^T(f)_p, \quad 0<t\le\frac{1}{r}. \tag{1.2}$$

Moreover, we have $\omega_r^T(f;t)_p\equiv 0$ if and only if $f\in T_{r-1}$. In that sense the new modulus of smoothness describes the rate of the best trigonometric approximation more precisely than the classical one. The modulus of smoothness $\omega_r^T(f;t)_p$ possesses properties similar to those of the classical one, as it is shown in [1].

Let $L_n : L_p^*[-\pi,\pi]\to L_p^*[-\pi,\pi]$, $1\le p<\infty$, or $L_n : C^*[-\pi,\pi]\to C^*[-\pi,\pi]$, be a bounded linear operator that preserves the trigonometric polynomials of degree n. Then the well-known Lebesgue inequality

$$\|f-L_nf\|_p \le (1+\|L_n\|)E_n^T(f)_p$$

and the Jackson-type estimate (1.1) imply

$$\|f-L_nf\|_p \le C_r(1+\|L_n\|)\,\omega_r^T(f,n^{-1})_p, \quad n\ge r-1.$$

Similar estimates, using the classical periodic modulus of smoothness, are known. For instance, G. P. Nevai has proved in [3] the following generalization of a result of S. M. Nikolskii:

$$\|f-t_nf\|_\infty \le 2^{-r}\omega_r\Big(f;\frac{2\pi}{2n+1}\Big)_\infty\lambda_n(\bar{x}) + \mathcal{O}\big(\omega_r(f;n^{-1})_\infty\big),$$

where $t_nf\in T_n$ interpolates $f\in C^*[-\pi,\pi]$ in the equidistant nodes $\bar{x}=(x_{-n},\dots,x_n)$, $x_k = 2k\pi/(2n+1)$, $k=-n,\dots,n$, and $\lambda_n(\bar{x})$ is the Lebesgue constant for the trigonometric Lagrange interpolation. For similar estimates in uniform norm, concerning the approximation by the partial sums of the Fourier series, one can refer to [2] and [4].

The trigonometric analogue of Taylor's formula will allow us to derive a pointwise estimate of the error $f(x)-L_n(f,x)$ for a smooth f. We need to introduce several notations to state that result. We define the differential operators

$$D_j = \Big(\frac{d}{dx}\Big)^2 + j^2I, \quad j=1,2,\dots, \tag{1.3}$$

where I is the identity. We also put

$$\begin{aligned}
\widetilde{D}_{n+1} &= D_n\cdots D_1\frac{d}{dx},\\
\widehat{D}_{n0} &= D_1\cdots D_n,\\
\widehat{D}_{nk} &= D_1\cdots D_{k-1}D_{k+1}\cdots D_n, \quad k=1,\dots,n.
\end{aligned}$$

Let us observe that $\widetilde{D}_{n+1}g = 0$, $g \in C^{2n+1}[a,b]$, if and only if $g \in T_n$ in $[a,b]$. The following trigonometric analogue of Taylor's formula holds true (see [5, §10.8]).

Theorem 1.1 (Taylor's trigonometric formula)**.** *Let $f \in C^{2n+1}(\Delta_c)$, where Δ_c is any of the intervals $[c, c+\delta]$, $[c-\delta, c]$ or $[c-\delta, c+\delta]$ for $c \in \mathbf{R}$ and $\delta > 0$, and let also*

$$(1.4) \quad \tau_{n,c}(f,x) = \frac{\widehat{D}_{n0}f(c)}{(n!)^2} + 2\sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n-k)!\,(n+k)!} \times \big[(k^2\widehat{D}_{nk}f(c) - \widehat{D}_{n0}f(c))\cos k(x-c) + k\widehat{D}_{nk}f'(c)\sin k(x-c)\big].$$

Then $\tau_{n,c}f \in T_n$, $\tau_{n,c}^{(s)}(f,c) = f^{(s)}(c)$, $s = 0, 1, \ldots, 2n$, and for $x \in \Delta_c$ we have

$$(1.5) \qquad f(x) = \tau_{n,c}(f,x) + \frac{1}{n!\,(2n-1)!!}\int_c^x (1-\cos(x-t))^n \widetilde{D}_{n+1}f(t)\,dt.$$

Let $-\pi \le x_0 < \cdots < x_{2n} < \pi$ be arbitrary nodes. Let us denote by $t_n(f,x)$ the unique trigonometric polynomial of degree n, which interpolates f in those nodes. Then the theorem above easily implies a point-wise estimate of the error $f(x) - t_n(f,x)$ for a smooth function f.

Proposition 1.2. *Let $f \in C^{2n+1}[-\pi,\pi]$. Then*

$$f(x) - t_n(f,x) = \frac{1}{n!\,(2n-1)!!}\int_{-\pi}^{\pi} K(x,t)\widetilde{D}_{n+1}f(t)\,dt, \quad x \in [-\pi,\pi],$$

where

$$K(x,t) = \big(1-\cos[(x-t)_+]\big)^n - \sum_{k=0}^{2n}\big(1-\cos[(x_k-t)_+]\big)^n t_{nk}(x)$$

and $(x-t)_+ = \max\{x-t, 0\}$.

The contents of the paper are organized as follows. In Section 2 we collect few auxiliary results, which are necessary for the proof of Taylor's trigonometric formula, presented in Section 3. Finally, in the last section we derive point-wise estimates of the error in the trigonometric interpolation and in the approximation by convolutional linear operators.

2 Auxiliary results

Let $[a,b]$ be a finite interval such that $0 \in [a,b]$. We define the convolutional operator, known as Duhamel's convolution, $\circledast : L_1[a,b] \times L_1[a,b] \to L_1[a,b]$,

$$f \circledast g(x) := \int_0^x f(x-t)g(t)\,dt.$$

It is easy to verify that it possesses the properties:

1. $f \circledast g = g \circledast f$;
2. $f \circledast (g+h) = f \circledast g + f \circledast h$;
3. $f \circledast (g \circledast h) = (f \circledast g) \circledast h$.

Next we introduce a number of notations. We put $\varphi_n(x) = \sin nx,\ n = 1, 2, \dots,$ and $\Phi_n = \varphi_1 \circledast \cdots \circledast \varphi_n$, $\widetilde{\Phi}_n = \Phi_n \circledast 1$, $\widehat{\Phi}_n = \widetilde{\Phi}_n \circledast 1$. The propositions bellow contain some of the properties of Φ_n, $\widetilde{\Phi}_n$ and $\widehat{\Phi}_n$, but first we prove the following simple lemma.

Lemma 2.1. *Any function of the form*

$$f(x) = cx + a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \tag{2.1}$$

has at most $2n+1$ *zeroes in* $[-\pi, \pi)$*, counting the multiplicities, that is,* $x, 1, \cos x, \sin x, \dots, \cos nx, \sin nx$ *is an extended Chebyshev system in* $[-\pi, \pi)$*. Hence, for any choice of* $-\pi \le x_1 < \cdots < x_m < \pi$ *and positive integers* $\nu_1, \dots, \nu_m$ *with* $\nu_1 + \cdots + \nu_m = 2n+1$ *there exists only one function of the form* (2.1) *with a fixed* c *for which* x_k *is a zero of multiplicity* $\nu_k,\ k = 1, \dots, m$.

Proof. It is enough to prove the first part of the statement. We follow a standard argument assuming the opposite and making use of the well-known Rolle's theorem. So let us assume that $f(x)$ has at least $2n+2$ zeroes in $[-\pi, \pi)$, counting the multiplicities. Then $f'(x)$ has at least $2n+1$ zeroes in $[-\pi, \pi)$, counting the multiplicities. But $f'(x)$ is a trigonometric polynomial of degree n and therefore it has at most $2n$ zeroes in $[-\pi, \pi)$, counting the multiplicities. This contradiction verifies the statement of the lemma. □

Proposition 2.2. *We have*

(i) $D_n \Phi_n = n\Phi_{n-1}$ *and* $D_n \widehat{\Phi}_n = n\widehat{\Phi}_{n-1}$ *for* $n = 2, 3, \dots$;

(ii) $\Phi_n(x) = c_n \sin x (1-\cos x)^{n-1}$*, where* $c_n = \dfrac{n}{(2n-1)!!}$, $n = 1, 2 \dots$;

(iii) $\widetilde{\Phi}_n(x) = \dfrac{1}{(2n-1)!!}(1-\cos x)^n$;

(iv) $\widehat{\Phi}_n(x) = \dfrac{x}{n!} + \sum_{k=1}^{n} b_{nk} \sin kx$,

where $\{b_{nk}\}$ *is the unique solution of the linear system*

$$\left| \begin{array}{l} \sum_{k=1}^{n} k b_{nk} = -\dfrac{1}{n!}, \\ \sum_{k=1}^{n} k^s b_{nk} = 0,\ s = 3, 5, \dots, 2n-1. \end{array} \right.$$

Proof. The first statement of the proposition follows by differentiation of the recursion relation $\Phi_n = \varphi_n \circledast \Phi_{n-1}$. Namely, we have

$$\begin{aligned}\left(\frac{d}{dx}\right)^2 \Phi_n(x) &= \left(\frac{d}{dx}\right)^2 \int_0^x \sin n(x-t)\Phi_{n-1}(t)\,dt = n\frac{d}{dx}\int_0^x \cos n(x-t)\Phi_{n-1}(t)\,dt \\ &= n\Phi_{n-1}(x) - n^2 \int_0^x \sin n(x-t)\Phi_{n-1}(t)\,dt \\ &= n\Phi_{n-1}(x) - n^2\Phi_n(x).\end{aligned}$$

Thus we have got $D_n\Phi_n = n\Phi_{n-1}$. If we put $e_1(x) = x$, then $\widehat{\Phi}_n = \widetilde{\Phi}_n \circledast 1 = \Phi_n \circledast 1 \circledast 1 = \Phi_n \circledast e_1$. Therefore $\widehat{\Phi}_n$ satisfies the same recursion relation as Φ_n with $\widehat{\Phi}_1(x) = x - \sin x$ instead of $\Phi_1(x) = \sin x$. Hence we get $D_n\widehat{\Phi}_n = n\widehat{\Phi}_{n-1}$. This completes the proof of (i).

To verify (ii), we consider the sequence of trigonometric polynomials

$$P_n(x) = c_n \sin x(1-\cos x)^{n-1},\ c_n = \frac{n}{(2n-1)!!},\ n \geq 1.$$

We shall show that it satisfies the same recursion relation as Φ_n in (i) and $P_n(0) = 0 = \Phi_n(0)$, $P_n'(0) = 0 = \Phi_n'(0)$, $n \geq 2$. Hence, as $P_1 = \Phi_1$, we have $P_n = \Phi_n$, $n \geq 2$. For $n \geq 2$

$$\begin{aligned}P_n''(x) &= c_n\big(\sin x(1-\cos x)^{n-1}\big)'' \\ &= c_n \sin x(1-\cos x)^{n-2}(n^2 - 3n + 1 + n^2\cos x).\end{aligned}$$

Consequently,

$$\begin{aligned}D_nP_n(x) &= P_n''(x) + n^2P_n(x) \\ &= c_n \sin x(1-\cos x)^{n-2}(n^2 - 3n + 1 + n^2\cos x) + n^2c_n \sin x(1-\cos x)^{n-1} \\ &= c_n(2n^2 - 3n + 1)\sin x(1-\cos x)^{n-2} \\ &= nc_{n-1}\sin x(1-\cos x)^{n-2} \\ &= nP_{n-1}(x).\end{aligned}$$

We get (iii) by integrating (ii).

It remains to verify (iv). From (i) it follows $\widetilde{D}_{n+1}\widehat{\Phi}_n = n!$. Consequently, $\widehat{\Phi}_n(x) = x/n! + a_0 + \sum_{k=1}^n (a_{nk}\cos kx + b_{nk}\sin kx)$ for some constants a_{nk}, b_{nk}. Assertion (iii) implies that $\widetilde{\Phi}_n$ is an even function, therefore $\widehat{\Phi}_n$ is an odd one. This implies that $\widehat{\Phi}_n(x) = x/n! + \sum_{k=1}^n b_{nk}\sin kx$ for some $b_{nk} \in \mathbf{R}$. Next we have $\widehat{\Phi}_n'(0) = \widetilde{\Phi}_n(0) = 0$, which implies

$$\sum_{k=1}^n kb_{nk} = -\frac{1}{n!}.$$

It is easy to see that $\widehat{\Phi}_n^{(s)}(0) = 0,\ s = 2, \ldots, 2n-1$, as well. For s even this is obvious. For s odd we can verify it, for instance, by induction in n. For $n = 1$

the statement is trivial as we have shown above. We assume that $\widehat{\Phi}_n^{(s)}(0) = 0,\ s = 1, \dots, 2n-1$, and shall verify it for $n+1$ in the place of n. We differentiate in x the equality $D_{n+1}\widehat{\Phi}_{n+1}(x) = (n+1)\widehat{\Phi}_n(x)$ and get for $s = 1, \dots, 2n-1$

$$\widehat{\Phi}_{n+1}^{(s+2)}(x) + (n+1)^2\widehat{\Phi}_{n+1}^{(s)}(x) = (n+1)\widehat{\Phi}_n^{(s)}(x).$$

Then, putting $x = 0$, we get $\widehat{\Phi}_{n+1}^{(s)}(0) = 0$ consecutively for $s = 3, 5, \dots, 2n+1$, which is what we had to show. Now

$$\sum_{k=1}^{n} k^s b_{nk} = 0,\ s = 3, 5, \dots, 2n-1,$$

follows from $\widehat{\Phi}_n^{(s)}(0) = 0,\ s = 3, \dots, 2n-1\ (n > 1)$. In passing, let us note that the linear system

$$\left| \begin{array}{l} \displaystyle\sum_{k=1}^{n} k b_{nk} = -\frac{1}{n!}, \\ \displaystyle\sum_{k=1}^{n} k^s b_{nk} = 0,\ s = 3, 5, \dots, 2n-1; \end{array} \right.$$

has a unique solution due to Lemma 2.1. This completes the proof of (iv). □

The following representation of $\widehat{\Phi}_n(x)$ has been pointed out to the author by K. G. Ivanov.

Proposition 2.3. *The following formula holds:*

$$\widehat{\Phi}_n(x) = \frac{1}{n!}\Big(x - \sum_{k=1}^{n} \frac{(k-1)!}{(2k-1)!!} \sin x (1 - \cos x)^{k-1}\Big). \tag{2.2}$$

Proof. We just write for $n \geq 1$

$$\begin{aligned} J_n(x) := \int_0^x \sin^{2n} t\, dt &= -\int_0^x \sin^{2n-1} t\, d\cos t \\ &= -\sin^{2n-1} x \cos x + (2n-1)\int_0^x \cos^2 t \sin^{2(n-1)} t\, dt \\ &= -\sin^{2n-1} x \cos x + (2n-1)\int_0^x \sin^{2(n-1)} t\, dt - (2n-1)\int_0^x \sin^{2n} t\, dt. \end{aligned}$$

Therefore

$$J_n(x) = -\sin^{2n-1} x \cos x + (2n-1)J_{n-1}(x) - (2n-1)J_n(x).$$

Hence we get the recursion relation

$$J_n(x) = -\frac{1}{2n}\sin^{2n-1} x \cos x + \frac{2n-1}{2n} J_{n-1}(x), \quad n \geq 1.$$

Consequently, noting that $J_0(x) = x$, we get

$$\begin{aligned} J_n(x) &= \frac{1}{2n}\Big(\frac{(2n-1)!!}{(2n-2)!!}x - \sin^{2n-1} x \cos x \\ &\qquad - \sum_{l=1}^{n-1} \frac{(2n-1)(2n-3)\cdots(2n-2l+1)}{(2n-2)(2n-4)\cdots(2n-2l)} \sin^{2n-2l-1} x \cos x\Big) \\ &= \frac{1}{2n}\Big(\frac{(2n-1)!!}{(2n-2)!!}x - \frac{1}{2}\sin^{2(n-1)} x \sin 2x \\ &\qquad - \frac{(2n-1)!!}{2(2n-2)!!}\sum_{l=1}^{n-1} \frac{(2n-2l-2)!!}{(2n-2l-1)!!} \sin^{2(n-l-1)} x \sin 2x\Big) \\ &= \frac{(2n-1)!!}{(2n)!!}\Big(x - \frac{(2n-2)!!}{2(2n-1)!!}\sin^{2(n-1)} x \sin 2x \\ &\qquad - \frac{1}{2}\sum_{l=1}^{n-1} \frac{(2n-2l-2)!!}{(2n-2l-1)!!} \sin^{2(n-l-1)} x \sin 2x\Big) \\ &= \frac{(2n-1)!!}{(2n)!!}\Big(x - \frac{1}{2}\sum_{l=0}^{n-1} \frac{(2n-2l-2)!!}{(2n-2l-1)!!} \sin^{2(n-l-1)} x \sin 2x\Big) \\ &= \frac{(2n-1)!!}{(2n)!!}\Big(x - \frac{1}{2}\sum_{k=1}^{n} \frac{(2k-2)!!}{(2k-1)!!} \sin^{2(k-1)} x \sin 2x\Big) \\ &= \frac{(2n-1)!!}{(2n)!!}\Big(x - \frac{1}{2}\sum_{k=1}^{n} \frac{(k-1)!}{(2k-1)!!} 2^{k-1} \sin^{2(k-1)} x \sin 2x\Big) \\ &= \frac{(2n-1)!!}{2^{n+1}n!}\Big(2x - \sum_{k=1}^{n} \frac{(k-1)!}{(2k-1)!!} (1-\cos 2x)^{k-1} \sin 2x\Big). \end{aligned}$$

Thus we have shown

$$J_n(x) = \frac{(2n-1)!!}{2^{n+1}n!}\Big(2x - \sum_{k=1}^{n} \frac{(k-1)!}{(2k-1)!!} \sin 2x (1-\cos 2x)^{k-1}\Big). \tag{2.3}$$

To finish the proof, we just write

$$\begin{aligned} \widehat{\Phi}_n(x) &= \frac{1}{(2n-1)!!}\int_0^x (1-\cos t)^n \, dt = \frac{2^{n+1}}{(2n-1)!!}\int_0^{x/2} \sin^{2n} t \, dt \\ &= \frac{2^{n+1}}{(2n-1)!!} J_n(x/2). \end{aligned}$$

Hence, making use of (2.3), we get (2.2). □

Let $[a,b]$ be a finite interval such that $0 \in [a,b]$. In [1] we have proved that $\mathcal{F}_n : C[a,b] \to C[a,b]$ can be represented in the form

$$\mathcal{F}_n = A_1 \cdots A_n,$$

where the bounded linear operators $A_j : C[a,b] \to C[a,b]$, $j = 1, 2, \dots,$ are defined by

$$A_j(f,x) := f(x) + j^2 \int_0^x (x-t) f(t)\, dt, \quad j = 1, 2, \dots$$

In the above mentioned investigation we have also shown the following assertion.

Proposition 2.4. *The bounded linear operator A_j is invertible and*

$$A_j^{-1}(g,x) = g(x) - j \int_0^x \sin j(x-t) g(t)\, dt.$$

Hence

$$A_j^{-1}(g,x) = \frac{1}{j} \int_0^x \sin j(x-t) g''(t)\, dt$$

for $g \in C^2[a,b]$ with $g(0) = g'(0) = 0$.

3 The proof of Taylor's trigonometric formula

Now we are ready to prove formula (1.5).

Proof of Theorem 1.1. It is enough to prove the assertion of the theorem for $c = 0$. Hence it will follow for any $c \in \mathbf{R}$ by translation. Let $\tau(x) = a_0 + a_1 \cos x + b_1 \sin x + \dots + a_n \cos nx + b_n \sin nx$ be the unique trigonometric polynomial of degree at most n, which interpolates f in $x = 0$ with multiplicity $2n+1$, i.e., $\tau^{(s)}(0) = f^{(s)}(0)$ for $s = 0, 1, \dots, 2n$. Using

$$D_j \cos kx = (j^2 - k^2) \cos kx \quad \text{and} \quad D_j \sin kx = (j^2 - k^2) \sin kx,$$

we get

$$\begin{aligned}
&(n!)^2 a_0 = \widehat{D}_{n0} \tau(0) = \widehat{D}_{n0} f(0),\\
&(-1)^{k-1} \frac{(n-k)!\,(n+k)!}{2k^2} a_k + \frac{(n!)^2}{k^2} a_0 = \widehat{D}_{nk} \tau(0) = \widehat{D}_{nk} f(0), \ k = 1, \dots, n,\\
&(-1)^{k-1} \frac{(n-k)!\,(n+k)!}{2k} b_k = \widehat{D}_{nk} \tau'(0) = \widehat{D}_{nk} f'(0), \ k = 1, \dots, n.
\end{aligned}$$

Hence $\tau_{n,0}(f,x) = \tau(x)$.

It remains to consider the remainder $r_n(x) = f(x) - \tau_{n,0}(f,x)$. Let us put for the sake of brevity

$$F(x) = \int_0^x \left(\int_0^{t_1} \cdots \left(\int_0^{t_{2n}} \widetilde{D}_{n+1} f(t_{2n+1})\, dt_{2n+1} \right) \cdots\, dt_2 \right) dt_1.$$

Obviously, $F \in C^{2n+1}(\Delta_0)$ and $F^{(s)}(0) = 0,\ s = 0, 1, \dots, 2n$. Now $r_n(x) = f(x) - \tau_{n,0}(f, x)$ implies $\widetilde{D}_{n+1} r_n(x) = \widetilde{D}_{n+1} f(x)$, $x \in \Delta_0$. We have proved in [1] that

$$(\mathcal{F}_n g)^{(2n+1)} = \widetilde{D}_{n+1} g, \quad g \in C^{2n+1}(\Delta_0).$$

Therefore $(d/dx)^{2n+1} \mathcal{F}_n(r_n, x) = \widetilde{D}_{n+1} r_n(x)$, $x \in \Delta_0$. Hence, making use of $r_n^{(s)}(0) = 0,\ s = 0, 1, \dots, 2n$, we get $\mathcal{F}_n(r_n, x) = F(x)$, $x \in \Delta_0$, that is,

$$A_1 \cdots A_n r_n = F. \tag{3.1}$$

Proposition 2.4 states for $g \in C^2(\Delta_0)$ with $g(0) = g'(0) = 0$ that

$$A_j^{-1} g = \frac{1}{j} \varphi_j \circledast g''. \tag{3.2}$$

Simple calculations yield for $g \in C^2(\Delta_0)$ with $g(0) = g'(0) = 0$

$$(\Phi_k \circledast g)'' = \Phi_k \circledast g'', \tag{3.3}$$

and for any $g \in C(\Delta_0)$

$$\Phi_k \circledast g(0) = (\Phi_k \circledast g)'(0) = 0. \tag{3.4}$$

Now (3.1) and (3.2) for $j = 1$ imply

$$A_2 \cdots A_n r_n = \varphi_1 \circledast F'' = \Phi_1 \circledast F''.$$

Next, applying again (3.2) (for $j = 2$), using (3.4) (for $k = 1$), and then (3.3) (for $k = 1$), we have

$$A_3 \cdots A_n r_n = \frac{1}{2} \varphi_1 \circledast \varphi_2 \circledast F^{(4)} = \frac{1}{2} \Phi_2 \circledast F^{(4)}.$$

Proceeding in this way, we finally get

$$r_n = \frac{1}{n!} \Phi_n \circledast F^{(2n)}. \tag{3.5}$$

To finish the proof, we write

$$\begin{aligned}
r_n(x) &= \frac{1}{n!} \int_0^x \left(\Phi_n(x - t) \int_0^t \widetilde{D}_{n+1} f(s)\, ds \right) dt \\
&= -\frac{1}{n!} \int_0^x \left(\int_0^t \widetilde{D}_{n+1} f(s)\, ds \right) d\widetilde{\Phi}_n(x - t) \\
&= \frac{1}{n!} \int_0^x \widetilde{\Phi}_n(x - t) \widetilde{D}_{n+1} f(t)\, dt \\
&= \frac{1}{n!} \widetilde{\Phi}_n \circledast \widetilde{D}_{n+1} f(x).
\end{aligned}$$

This completes the proof of the theorem as Proposition 2.2 (iii) states $\widetilde{\Phi}_n(x) = 1/(2n-1)!!\,(1 - \cos x)^n$. □

Remark 3.1. An estimate of the remainder. (Again we discuss the case $c = 0$.) The mean value theorem implies

$$r_n(x) = \frac{\widetilde{D}_{n+1} f(\xi_x)}{n!\,(2n-1)!!} \int_0^x (1-\cos t)^n \, dt, \; x \in \Delta_0, \tag{3.6}$$

where $\xi_x \in \Delta_0$ depends on x. Hence

$$|r_n(x)| \le \frac{\|\widetilde{D}_{n+1} f\|_{\infty(\Delta_0)}}{n!\,(2n-1)!!} \left| \int_0^x (1-\cos t)^n \, dt \right|, \; x \in \Delta_0. \tag{3.7}$$

Now, using the simple inequality $1 - \cos x \le x^2/2$, we get

$$|r_n(x)| \le \frac{|x|^{2n+1}}{(2n+1)!} \|\widetilde{D}_{n+1} f\|_{\infty(\Delta_0)}, \; x \in \Delta_0. \tag{3.8}$$

4 Application

Formula (1.5) can be useful in expressing the error in approximation by linear operators that preserves trigonometric polynomials up to a given degree. Indeed, let $L_n : C[-\pi, \pi] \to C[-\pi, \pi]$ be such that $L_n f = f$ if $f \in T_n$ and let $f \in C^{2n+1}[-\pi, \pi]$. Then we have

$$f - L_n f = (I - L_n) r_n f, \tag{4.1}$$

where

$$r_n(f, x) = \frac{1}{n!\,(2n-1)!!} \int_c^x (1 - \cos(x-t))^n \widetilde{D}_{n+1} f(t) \, dt$$

for some fixed $c \in [-\pi, \pi]$.

Let $-\pi \le x_0 < \cdots < x_{2n} < \pi$ be arbitrary nodes. Then, as it is known, there exists a unique trigonometric polynomial $t_n(f, x)$ of degree n such that $t_n(f, x_k) = f(x_k), \; k = 0, \ldots, 2n$. It can be represented in the form

$$t_n(f, x) = \sum_{k=0}^{2n} f(x_k) t_{nk}(x), \tag{4.2}$$

where

$$t_{nk}(x) = \frac{\displaystyle\prod_{j=0, j\ne k}^{2n} \sin \frac{x - x_j}{2}}{\displaystyle\prod_{j=0, j\ne k}^{2n} \sin \frac{x_k - x_j}{2}}. \tag{4.3}$$

Now the considerations in the beginning of this section and (1.5) with $c = -\pi$ easily yield Proposition 1.2. That proposition implies the following estimates of the error $f(x) - t_n(f, x)$ for smooth functions f.

Corollary 4.1. *Let $f \in C^{2n+1}[-\pi,\pi]$. Then we have for $x \in [-\pi,\pi]$*

(i) $|f(x) - t_n(f,x)| \le \dfrac{\pi\mu(\bar{x})\|\widetilde{D}_{n+1}f\|_\infty}{2^n(n-1)!\,(2n-1)!!}\,|(x-x_0)\ldots(x-x_{2n})|,$

where

$$\mu(\bar{x}) = \sum_{k=0}^{2n}\left(\prod_{j=0,j\neq k}^{2n}\left|\sin\frac{x_k - x_j}{2}\right|\right)^{-1}.$$

(ii) $|f(x) - t_n(f,x)| \le \dfrac{2^{n+1}\pi^2\mu(\bar{x})\|\widetilde{D}_{n+1}f\|_\infty}{a\,(n-1)!\,(2n-1)!!}\left|\sin\dfrac{x-x_0}{2}\cdots\sin\dfrac{x-x_{2n}}{2}\right|$

for nodes $-\pi + a \le x_0 < \cdots < x_{2n} \le \pi - a,\ a \in (0,\pi)$.

Proof. The assertions of the corollary follow easily from the estimate

$$(4.4)\quad \big|(1-\cos[(x-t)_+])^n - (1-\cos[(x_k-t)_+])^n\big| \le n2^{n-1}|\cos[(x_k-t)_+] - \cos[(x-t)_+]|$$

and the relation

$$(4.5)\quad \cos[(x_k-t)_+]-\cos[(x-t)_+] = \begin{cases} -2\sin\dfrac{x+x_k-2t}{2}\sin\dfrac{x_k-x}{2}, & t \le x, x_k,\\ 2\sin^2\dfrac{x-t}{2}, & x_k \le t \le x,\\ -2\sin^2\dfrac{x_k-t}{2}, & x \le t \le x_k,\\ 0, & t \ge x, x_k. \end{cases}$$

Now (4.4), (4.5) and the inequality $|\sin x| \le |x|$ imply

$$|(1-\cos[(x-t)_+])^n - (\cos[(x_k-t)_+])^n| \le n2^{n-1}|x-x_k|,$$

therefore, using again the inequality $|\sin x| \le |x|$ and the fact that $\sum_{k=0}^{2n} t_{nk}(x) \equiv 1$, we get for any x and t

$$|K(x,t)| \le n2^{n-1}\sum_{k=0}^{2n}|x-x_k||t_{nk}(x)| \le n2^{-n-1}\mu(\bar{x})|x-x_0|\cdots|x-x_{2n}|.$$

Hence assertion (i) follows. To verify, (ii) we just have to notice that if $-\pi + a \le x_0 < \cdots < x_{2n} \le \pi - a$, where $a \in (0,\pi)$, and $x \in [-\pi,\pi]$, then

$$\sin\frac{x-t}{2} \le \frac{\sin\dfrac{x-x_k}{2}}{\sin\dfrac{a}{2}},\ x_k \le t \le x, \quad \text{and} \quad \sin\frac{x_k-t}{2} \le \frac{\sin\dfrac{x_k-x}{2}}{\sin\dfrac{a}{2}},\ x \le t \le x_k.$$

These two estimates, (4.4), (4.5) and the inequality $|\sin x| \geq (2/\pi)|x|$, $|x| \leq \pi/2$, yield for $x \in [-\pi, \pi]$ and any t

$$|(1-\cos[(x-t)_+])^n - (\cos[(x_k-t)_+])^n| \leq \frac{n2^n\pi}{a}\left|\sin\frac{x-x_k}{2}\right|,$$

which, on its turn, implies for $x \in [-\pi, \pi]$ and any t

$$\begin{aligned} |K(x,t)| &\leq \frac{n2^n\pi}{a}\sum_{k=0}^{2n}\left|\sin\frac{x-x_k}{2}\right||t_{nk}(x)| \\ &= \frac{n2^n\pi\mu(\bar{x})}{a}\left|\sin\frac{x-x_0}{2}\right|\cdots\left|\sin\frac{x-x_{2n}}{2}\right|. \end{aligned}$$

Hence assertion (ii) follows. □

Remark 4.2. Our conjecture is that for any fixed $x' \in [-\pi, \pi]$ the kernel $K(x', t)$ does not change its sign in $[-\pi, \pi]$. If that is true, then the mean value theorem implies the Lagrange-type estimate

$$f(x) - t_n(f,x) = \frac{\widetilde{D}_{n+1}f(\xi_x)}{(n!)^2}\omega(x), \quad x \in [-\pi, \pi],$$

where $f \in C^{2n+1}[-\pi, \pi]$, and

$$\omega(x) = x + a_0 + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)$$

is the only function of this form, which vanishes in the nodes $\{x_k\}_{k=0}^{2n}$ and has no other zeroes in $[-\pi, \pi)$. Actually,

$$\omega(x) = x - \sum_{k=0}^{2n} x_k t_{nk}(x).$$

Let the bounded linear operator $L_n : C^*[-\pi, \pi] \to C^*[-\pi, \pi]$ be of the form

$$L_n(f,x) = \mathcal{M}_n * f(x) := \int_{-\pi}^{\pi}\mathcal{M}_n(x-t)f(t)\,dt, \tag{4.6}$$

where $\mathcal{M}_n \in L_1^*[-\pi, \pi]$. For any fixed $t \in [-\pi, \pi]$ we define the 2π-periodic function $\rho_t : \mathbf{R} \to \mathbf{R}$ by

$$\rho_t(x) := 1 - \cos[(x - 2k\pi - t)_+], \quad x \in [(2k-1)\pi, (2k+1)\pi),\ k \in \mathbf{Z}.$$

It is quite easy to verify the following assertion.

Proposition 4.3. *Let $f \in C^{2n+1}[-\pi,\pi]$ be 2π-periodic. Let also the bounded linear operator L_n, defined by* (4.6)*, preserve the trigonometric polynomials of degree n. Then*

$$f(x) - L_n(f,x) = \frac{1}{n!\,(2n-1)!!}\int_{-\pi}^{\pi} [\rho_t^n(x) - \mathcal{M}_n * \rho_t^n(x)]\widetilde{D}_{n+1}f(t)\,dt.$$

Proof. Making use of formula (1.5) with $c = -\pi$ and changing the order of integration after that, we get easily the estimate

$$\begin{aligned} f(x) - L_n(f,x) &= \frac{1}{n!\,(2n-1)!!}\int_{-\pi}^{\pi}(1-\cos[(x-t)_+])^n \widetilde{D}_{n+1}f(t)\,dt \\ &- \frac{1}{n!\,(2n-1)!!}\int_{-\pi}^{\pi}\mathcal{M}_n(x-t)\left(\int_{-\pi}^{\pi}(1-\cos[(t-u)_+])^n\widetilde{D}_{n+1}f(u)\,du\right)dt \\ &= \frac{1}{n!\,(2n-1)!!}\int_{-\pi}^{\pi}\Big((1-\cos[(x-t)_+])^n \\ &- \int_{-\pi}^{\pi}\mathcal{M}_n(x-u)(1-\cos[(u-t)_+])^n\,du\Big)\widetilde{D}_{n+1}f(t)\,dt. \end{aligned}$$

Thus the proof is completed. □

Immediately, Proposition 4.3 yields

Corollary 4.4. *Let $f \in C^{2n+1}[-\pi,\pi]$ be 2π-periodic. Let also the bounded linear operator L_n, defined by* (4.6)*, preserve the trigonometric polynomials of degree n. Then*

$$\|f - L_n f\|_\infty \le \frac{2^{n+1}\pi}{n!\,(2n-1)!!}(1 + \|\mathcal{M}_n\|_1)\|\widetilde{D}_{n+1}f\|_\infty.$$

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