A new modulus of smoothness for trigonometric polynomial approximation

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Abstract. The rate of convergence of best trigonometric approximation in L_p and C-norm is characterized by a new modulus of smoothness. This modulus is equivalent to 0 on the trigonometric polynomials up to a given degree.

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1 Introduction and main results

The approximation by trigonometric polynomials is well investigated. The rate of convergence in uniform and integral norm was described by the classical modulus of smoothness due to D. Jackson, S. N. Bernstein, A. Zygmund and S. B. Stechkin (see for example [4]). This result was extended to any Banach space of 2π-periodic functions for which translation is continuous isometry by H. S. Shapiro and Z. Ditzian ([5] and [2]).

Let B be a homogeneous Banach space of 2π-periodic real-valued functions. We denote by T_n the set of all trigonometric polynomials of degree at most n and put

$$E_n^T(f)_B = \inf_{\tau \in T_n \cap B} \|f - \tau\|_B$$

for the best trigonometric approximation. It was shown by the authors mentioned above that for $f \in B$ we have

$$\begin{aligned} &E_n^T(f)_B \le C_r \omega_r(f; n^{-1})_B, \\ &\omega_r(f;t)_B \le C_r t^r \sum_{0 \le k \le 1/t} (k+1)^{r-1} E_k^T(f)_B, \ 0 < t \le t_0, \end{aligned}$$

where $\omega_r(f;t)_B$ is the classical modulus of smoothness defined as follows

$$\begin{aligned} \omega_r(f;t)_B &= \sup_{0<h\le t} \|\Delta_h^r f\|_B\,, \\ \Delta_h^r f(x) &= \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r/2 - k)h). \end{aligned}$$

Let B denote $L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $C^*[-\pi,\pi]$ where

$$L_p^*[-\pi,\pi] = \{f : \mathbf{R} \to \mathbf{R} : f(x+2\pi) = f(x) \text{ a.e., } f|_{[-\pi,\pi]} \in L_p[-\pi,\pi]\}$$
$$C^*[-\pi,\pi] = \{f \in C(\mathbf{R}) : f(x+2\pi) = f(x)\}.$$

It is interesting to construct a modulus-like function $\omega_r^T(f;t)_p$ associated with best trigonometric approximation such that for $f \in B$ we have

$$\omega_r^T(f;t)_p \equiv 0 \Longleftrightarrow f \in T_{r-1} \tag{1.1}$$

(here and further $f \in T_{r-1}$ in L_p-spaces means that f coincides a.e. with a trigonometric polynomial of degree at most $r-1$; we use that agreement in similar cases as well) and $\omega_r^T(f;t)_p$ characterizes the best trigonometric approximation like the classical modulus does. Thus this new modulus of smoothness describes more precisely (in the sense of (1.1)) the rate of convergence of best trigonometric approximation than the classical one. The definition of this new modulus, as we should expect, is more complicated. We shall show that the modulus of smoothness defined by

$$\omega_r^T(f;t)_p = \sup_{0<h\le t} \|\Delta_h^{2r-1} \mathcal{F}_{r-1} f\|_p,\ r = 1,2,\dots, \tag{1.2}$$

where

$$\mathcal{F}_{r-1}(f,x) = f(x) + \int_0^x \mathcal{K}_{r-1}(t) f(x-t)\,dt \tag{1.3}$$

and

$$\mathcal{K}_{r-1}(t) = \sum_{j=1}^{r-1} \frac{a_j^{(r-1)}}{(2j-1)!} t^{2j-1}, \quad a_j^{(r-1)} = \sum_{1\le l_1<\cdots<l_j\le r-1} (l_1 \cdots l_j)^2,$$

satisfies (1.1) and the following theorem

Theorem 1.1. *Let $f \in B$ where $B = L_p^*[-\pi,\pi],\ 1 \le p < \infty$, or $B = C^*[-\pi,\pi]$. Then*

$$E_n^T(f)_p \le C_r \omega_r^T(f;n^{-1})_p,\ n \ge r-1, \tag{1.4}$$

and

$$\omega_r^T(f;t)_p \le C_r t^{2r-1} \sum_{r-1\le k\le 1/t} (k+1)^{2r-2} E_k^T(f)_p,\ 0 < t \le \frac{1}{r}. \tag{1.5}$$

In particular (1.4) gives the trigonometric analogue of Whitney theorem [6], as it was observed by Bl. Sendov.

Theorem 1.2. *Let $f \in B$ where $B = L_p^*[-\pi,\pi],\ 1 \le p < \infty$, or $B = C^*[-\pi,\pi]$. Then*

$$E_{n-1}^T(f)_p \le C_n \omega_n^T(f;n^{-1})_p.$$

Let us observe that although $\mathcal{F}_{r-1}f$ is not generally a 2π-periodic function $\Delta_h^{2r-1}\mathcal{F}_{r-1}f$ is. Unlike the various moduli which describe the best algebraic approximation $\omega_r^T(f;t)_p$ is based on finite differences only of an odd order thus $\omega_r^T(f;t)_p$ is connected with the $(2r-1)$th finite difference not the rth one. This is due to the dimensions of the spaces T_{r-1} and Π_{r-1}, respectively. To state our next main result we define the K-functional

$$K_r^T(f;t)_p = \inf_{g\in B^{2r-1}}\{\|f-g\|_p + t^{2r-1}\|\widetilde{D}_r g\|_p\} \tag{1.6}$$

where we have put $B^s = \{g \in B : g, g', \ldots, g^{(s-1)} \in AC^*[-\pi,\pi], g^{(s)} \in B\}$, $AC^*[-\pi,\pi]$ being the set of all 2π-periodic absolutely continuous functions, $\widetilde{D}_r g = D_{r-1}\cdots D_1 g'$ and $D_j g = g'' + j^2 g$. We have $\widetilde{D}_r g = 0$ if and only if $g \in T_{r-1}$.

We write $\varphi(f;t) \sim \psi(f;t)$ if and only if there exists a constant $C > 0$ independent of f and t such that $C^{-1}\varphi(f;t) \le \psi(f;t) \le C\varphi(f;t)$. The following result holds

Theorem 1.3. *For $f \in B$ where $B = L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $B = C^*[-\pi,\pi]$ we have*

$$K_r^T(f;t)_p \sim \omega_r^T(f;t)_p$$

where $\omega_r^T(f;t)_p$ and $K_r^T(f;t)_p$ are defined in (1.2) *and* (1.6)*, respectively.*

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2 A new periodic modulus of smoothness

Let $[a,b]$ be an arbitrary finite subinterval of the real line such that $0 \in [a,b]$. We write $X = X[a,b]$ for any of the functional spaces $L_p[a,b]$, $1 \le p < \infty$, or $C[a,b]$ and $X^r = X^r[a,b]$ for the Sobolev spaces $W_p^r[a,b]$, $1 \le p < \infty$, or $C^r[a,b]$. We also write B for $L_p^*[-\pi,\pi]$, $1 \le p < \infty$, or $C^*[-\pi,\pi]$. We write B^r for $W_p^{*r}[-\pi,\pi]$, $1 \le p < \infty$, or $C^{*r}[-\pi,\pi]$, where

$$W_p^{*r}[-\pi,\pi] = \{f \in L_p^*[-\pi,\pi] : f, f', \ldots, f^{(r-1)} \in AC^*[-\pi,\pi], f^{(r)} \in L_p^*[-\pi,\pi]\},$$
$$C^{*r}[-\pi,\pi] = \{f \in C^*[-\pi,\pi] : f^{(k)} \in C^*[-\pi,\pi],\ k = 1,\ldots,r\}.$$

The proof of the main result is based on several auxiliary ones. First we consider the inverse of the bounded operator $\mathcal{F}_r : X \to X$.

Proposition 2.1. *We have*

$$\mathcal{F}_r^{-1}(g,x) = g(x) - \int_0^x \mathcal{L}_r(t)g(x-t)\,dt$$

where

$$\mathcal{L}_r(t) = 2\sum_{j=1}^r \frac{j^{2r}}{\omega_r'(j)}\sin jt \tag{2.1}$$

and $\omega_r(x) = \prod_{k=1}^{r}(x^2 - k^2)$. *Hence* $\mathcal{F}_r^{-1} : X \to X$ *is bounded.*

Formula (2.1) was pointed to the author by K. G. Ivanov. Let Π_s denote the set of all algebraic polynomials of degree at most s.

Proposition 2.2. *For* $\mathcal{F}_r$ *defined in* (1.3)*, we have*

(a) $(\mathcal{F}_r g)^{(2r+1)} = \widetilde{D}_{r+1} g,\ g \in X^{2r+1}$;

(b) $\mathcal{F}_r \tau \in \Pi_{2r},\ \tau \in T_r$;

(c) $\mathcal{F}_r^{-1} P \in T_r,\ P \in \Pi_{2r}$.

After these preliminaries we can formulate the main result of the section. First we introduce the bounded linear operator $\widetilde{\mathcal{F}}_{r-1} : B \to B$, $r \geq 2$

$$\widetilde{\mathcal{F}}_{r-1}(f, x) = \mathcal{F}_{r-1}(f, x) + P_{2r-2}(f, x)$$

where $\mathcal{F}_{r-1}$ is defined in (1.3) and $P_{2r-2}(f, x) = -\sum_{k=1}^{2r-2} \alpha_k (x + \pi)^k / k!$ is the unique algebraic polynomial of degree $2r - 2$ such that we may have $(\widetilde{\mathcal{F}}_{r-1} f)^{(s)}(-\pi) = (\widetilde{\mathcal{F}}_{r-1} f)^{(s)}(\pi)$, $s = 0, 1, \ldots, 2r - 3$, for any $f \in B^{2r-3}$ and hence $\widetilde{\mathcal{F}}_{r-1} : B^s \to B^s$, $s \in \mathbf{N}$.

Let I be the identity. For consistency we put $\mathcal{F}_0 = I$ ($\mathcal{K}_0 = 0$) and $P_0 = 0$ and then $\widetilde{\mathcal{F}}_0 = I$.

We introduce the following modulus of smoothness for a function $f \in B$ and $t > 0$:

$$\omega_r^T(f; t)_p = \omega_{2r-1}(\widetilde{\mathcal{F}}_{r-1} f; t)_p,\ r = 1, 2, \ldots$$

where $\omega_{2r-1}(F; t)_p$ is the classical periodic modulus of smoothness of order $2r - 1$. Let us note that $\Delta_h^{2r-1} \mathcal{F}_{r-1} f \in B$ for any $f \in B$ and

$$\omega_r^T(f; t)_p = \sup_{0<h\leq t} \|\Delta_h^{2r-1} \mathcal{F}_{r-1} f\|_p,\ r = 1, 2, \ldots .$$

In the definition of $\omega_r^T(f; t)_p$, $\Delta_h^{2r-1} \widetilde{\mathcal{F}}_{r-1} f(x)$ depends only on the values of f in a neighbourhood of x whose diameter diminishes with h. This can be seen in the examples at the end of the section. The point 0 in the integration limits of the integral operator used in the definition of $\widetilde{\mathcal{F}}_{r-1}$ has been chosen only for convenience – any other value can be fixed and the definition of $\omega_r^T(f; t)_p$ is invariant of this choice.

Now (1.1) follows immediately from Proposition 2.2 (b)-(c) and the definition of $\omega_r^T(f; t)_p$.

The following equivalence result holds.

Theorem 2.3. *For* $f \in B$ *where* $B = L_p^*[-\pi, \pi]$, $1 \leq p < \infty$, *or* $B = C^*[-\pi, \pi]$ *and any* $l \in \mathbf{N}$ *we have*

$$\inf_{g \in B^{2r+l-1}} \left\{ \|f - g\|_p + t^{2r+l-1} \|(d/dx)^l \widetilde{D}_r g\|_p \right\} \sim \omega_{2r+l-1}(\widetilde{\mathcal{F}}_{r-1} f; t)_p.$$

The proof of this theorem is based on the properties of the operator $\widetilde{\mathcal{F}}_{r-1}$ (see the propositions above) and those of another bounded linear operator $\mathcal{E}_{r-1} : B \to B$ defined by $\mathcal{E}_{r-1}(F,x) = \mathcal{F}_{r-1}^{-1}(F,x) + Q_{2r}(F,x)$ where $Q_{2r}(F,x)$ is an algebraic polynomial of the form $Q_{2r}(F,x) = \sum_{k=1}^{2r} \beta_k (x+\pi)^k/k!$, depending on F, such that $\mathcal{E}_{r-1}F \in B^s$ for any $F \in B^s$ for $s = 0,1,\ldots,2r-1$.

Now Theorem 1.3 follows from Theorem 2.3 for $l = 0$ and the definition of $\omega_r^T(f;t)_p$.

It is easy to verify, using the definition of $\omega_r^T(f;t)_p$ and some properties of the operator $\widetilde{\mathcal{F}}_{r-1}$, that $\omega_r^T(f;t)_p$ possesses the properties:

1. $\omega_r^T(f+g;t)_p \le \omega_r^T(f;t)_p + \omega_r^T(g;t)_p$ for $f,g \in B$.
2. $\omega_r^T(cf;t)_p = |c|\,\omega_r^T(f;t)_p,\ c$ is a constant.
3. $\omega_r^T(f;t)_p \le \omega_r^T(f;t')_p,\ t \le t'$.
4. $\omega_r^T(f;t)_p \to 0$ as $t \to 0$.
5. $\omega_r^T(f;t)_p \le (4+(r-1)^2t^2)\,\omega_{r-1}^T(f;t)_p,\ r \ge 2$.
6. $\omega_1^T(f;t)_p \le 2\|f\|_p$ and $\omega_1^T(f;t)_p \le t\|f'\|_p,\ f \in B^1$ ($\omega_1^T(f;t)_p$ coincides with the ordinary modulus of continuity).
7. $\omega_r^T(f;\lambda t)_p \le (\lambda+1)^{2r-1}\omega_r^T(f;t)_p$.
8. $\omega_r^T(f;t)_p \le t^2\omega_{r-1}^T(D_{r-1}f;t)_p,\ f \in B^2;\ r \ge 2$.
9. Marchaud inequality
$$\omega_r^T(f;t)_p \le C_r t^{2r-1}\left(\int_t^c \frac{\omega_{r+1}^T(f;u)_p}{u^{2r}}\,du + \|f\|_p\right),\ 0 < t \le c$$
where c is any fixed positive constant.
10. $\omega_r^T(f;t)_p = o(t^{2r-1}) \implies f \in T_{r-1}$ and $f \in T_{r-1} \implies \omega_r^T(f;t)_p \equiv 0$, $1 \le p \le \infty$.
11. $\omega_r^T(f;t)_p = \mathcal{O}(t^{2r-1}) \iff f \in W_p^{*2r-1}[-\pi,\pi]$, $1 < p \le \infty$.
12. $\omega_r^T(f;t)_1 = \mathcal{O}(t^{2r-1}) \iff f^{(2r-3)} \in AC^*[-\pi,\pi]$, $f^{(2r-2)} \in BV[-\pi,\pi]$.

At the end of this section we give the explicit form of the differential operator $\widetilde{D}_r$ as well as that of $\mathcal{K}_{r-1}(t)$ and $\Delta_h^{2r-1}\widetilde{\mathcal{F}}_{r-1}f(x)$ for $r = 1,2$.

$$\widetilde{D}_1 g = g', \qquad \mathcal{K}_0(t) = 0,$$
$$\widetilde{D}_2 g = g''' + g'; \qquad \mathcal{K}_1(t) = t;$$

$$\Delta_h^1\widetilde{\mathcal{F}}_0 f(x) = \Delta_h^1 f(x),$$
$$\Delta_h^3\widetilde{\mathcal{F}}_1 f(x) = \Delta_h^3 f(x) + \int_{-\frac{3}{2}h}^{\frac{3}{2}h} k_{1,h}(t) f(x-t)\,dt,$$

where

$$k_{1,h}(t) = \begin{cases} t + \frac{3}{2}h, & t \in [-\frac{3}{2}h, -\frac{1}{2}h], \\ -2t, & t \in [-\frac{1}{2}h, \frac{1}{2}h], \\ t - \frac{3}{2}h, & t \in [\frac{1}{2}h, \frac{3}{2}h], \\ 0, & \text{otherwise}. \end{cases}$$

3 Best trigonometric approximation

Now we can sketch the proof of our main result concerning the rate of best trigonometric approximation in $L_p^*[-\pi, \pi]$, $1 \le p < \infty$, and $C^*[-\pi, \pi]$.

Sketch of the proof of Theorem 1.1. To prove (1.4) we follow some methods demonstrated in [1] and [3] and use the modified Riesz operator

$$L_{r-1,n} = I - \prod_{j=0}^{r-1}(I - R_{j,n}) = \sum_{i=0}^{r-1}(-1)^i \sum_{0 \le j_0 < \cdots < j_i \le r-1} R_{j_0,n} \cdots R_{j_i,n}$$

where I is the identity and

$$R_{j,n}(f, x) = \sum_{k=0}^{n-1}\Big(1 - \frac{k^2 - j^2}{n^2 - j^2}\Big)A_k(x), \; n > j, \; j = 1, 2, \ldots,$$

$A_k(x) = A_k(f, x)$ being the k^{th} term in the Fourier expansion of f.

The inequality (1.5) can be proven similarly to its classical analogue. $\square$

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