# An Estimation of the Best Monotone Spline Approximation with the Averaged Moduli of Smoothness

P. E. Parvanov

05.11.1998

**Abstract**

In this paper a Jackson-type estimation for the approximation of monotone nondecreasing funcion $f$ by monotone nondecreasing splines with equally spaced knots in the $L_p[0,1]$-norm $(1 \le p \le \infty)$ is obtained. The estimation involves high order Sendov-Popov averaged moduli of smoothness of the derivative of $f$ and are obtained for function $f$ with a bounded and measurable derivative. The Chui, Smith and Ward's technique is used. The result is a generalization of the results in [2].

## 1 Introduction.

For $1 \le p < \infty$ let $L_p[0,1]$ denote the space of measurable functions whose $p$-th power is integrable and let $L_\infty[0,1]$ denote the space of bounded and measurable functions. Given $f \in L_p[0,1]$, define its $r$-th $L_p$-modulus of smoothness by

$$\omega_r(f,h)_{p[0,1]} \stackrel{\text{def}}{=} \sup\left\{ \|\Delta^r_{t,[0,1]} f(\cdot)\|_{p[0,1]} \ ; \ 0 \le t \le h \right\}$$

where

$$\Delta^r_{t,[0,1]} f(x) \stackrel{\text{def}}{=} \begin{cases} \sum_{i=0}^r (-1)^{r-i} \binom{r}{i} f(x+it) & if \ \ x, x+rt \in [0,1]; \\ 0 & otherwise. \end{cases}$$

Let $S(r,n)(r \ge 1)$ denote the space of all splines of order $r$ on the $n+1$ equally spaced knots $\left\{\frac{i}{n}\right\}_{i=0}^n$, i.e. $s \in S(r,n)$, if $s$ is a polynomial of degree $\le r-1$ in each interval $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ and $s^{(r-2)}$ is continuous in $[0,1]$. For $r=1$ $s$ is a piecewise constant function without continuity at the knots.

If $f \in L_p[0,1]$ is monotone nondecreasing, denote

$$E_n^{\uparrow}(f,r)_{p[0,1]} \stackrel{\text{def}}{=} \inf\left\{ \|f-s\|_{p[0,1]} \ ; \ s \in S(r,n) \ , \ s \ \ nondecreasing \right\}.$$

The following two theorems was proved by Leviatan-Mhaskar [2]

**Theorem 1** *If f possesses a continuous nonnegative derivative $f'$ on [0,1], then there is a constant c(r) which depends only of $r \geq 2$ such that*

$$E_n^{\uparrow}(f,r)_{\infty[0,1]} \leq c(r)n^{-1}\omega_{r-1}(f',n^{-1})_{\infty[0,1]}.$$

**Theorem 2** *Let $1 \leq p < \infty$. If f is the second primitive of $f'' \in L_p[0,1]$ and f is nondecreasing , then there is a constant c(r) which depends only of $r \geq 3$ such that*

$$E_n^{\uparrow}(f,r)_{p[0,1]} \leq c(r)n^{-2}\omega_{r-2}(f'',n^{-1})_{p[0,1]}.$$

For a function $f$ bounded on [0,1] the local modulus of smoothness of order $r$ at the point $x \in [0,1]$ is the function (see Definition 1.4 of [3])

$$\omega_r(f,x;\delta) \stackrel{\text{def}}{=} \sup\left\{|\Delta_{h,[0,1]}^r f(t)| \ ; \ t, t+rh \in \left[x - \frac{r\delta}{2}, x + \frac{r\delta}{2}\right]\right\}$$

For $1 \leq p \leq \infty$ the $r$-th order averaged Sendov-Popov modulus of smoothness of a function $f$ bounded and measurable on [0,1] is (see Definition 1.5 of [3])

$$\tau_r(f,\delta)_{p[0,1]} \stackrel{\text{def}}{=} \|\omega_r(f,\cdot;\delta)\|_{p[0,1]}.$$

The following properties of $\tau_r$ are used (see Theorem 1.5 and Property 5 of [3]). Let $1 \leq p \leq \infty$ and $f$ is the primitive of $f' \in L_p[0,1]$ then there is a constant $c(r)$ which depends only of $r \geq 2$ such that

$$\tau_r(f,\delta)_{p[0,1]} \leq c(r)\delta\omega_{r-1}(f',\delta)_{p[0,1]}. \tag{1}$$

Let $f$ be measurable on $[0,1]$ and $k$ is integer. Then

$$\tau_r(f,k\delta)_{p[0,1]} \leq k^{r+1}\tau_r(f,\delta)_{p[0,1]}. \tag{2}$$

The main result of this paper is the following stronger estimation of best monotone spline approximation.

**Theorem 3** *Let $1 \leq p \leq \infty$. If f is the primitive of a bounded and measurable on [0,1] function $f'$ and f is nondecreasing, then there is a constant c(r) which depends only of $r \geq 2$ such that*

$$E_n^{\uparrow}(f,r)_{p[0,1]} \leq c(r)n^{-1}\tau_{r-1}(f',n^{-1})_{p[0,1]}.$$

**Remark 1** For $p = \infty$ Theorem 3 coincides with Theorem 1 because of $\tau_r(f,\delta)_{\infty[0,1]} \equiv \omega_r(f,\delta)_{\infty[0,1]}$.
**Remark 2** Theorem 2 follows from Theorem 3 because of (1).

In order to prove the main result we use some statements from [2] .

**Lemma 1** *Let f be continuously differentiable on [-1,1] and nondecreasing there. Then there is nondecreasing polynomial P on [-1,1] of degree $\leq r$ $(r \geq 1)$ which interpolates f at 0 and 1 and such that*

$$\|f-P\|_{\infty[-1,1]} \le c(r)\,\omega_r(f',1)_{\infty[-1,1]}.$$

This is Lemma 3.2(i) from [2].

**Remark 3** This statement is valid for a nondecreasing function $f$ which is the primitive of a bounded and measurable function $f'$ (see the proof of Lemma 3.2(i) from [2]).

**Lemma 2** *Let f be a nondecreasing function which is the primitive of a bounded and measurable on $[-1,1]$ function $f'$. For $r \ge 1$ there exists a nondecreasing continuous function $g$ on [-1,1] such that $g$ interpolates $f$ at -1, 0 and 1 and has the properties:*

(*i*) *The restrictions of $g$ to $[-1,0]$ and $[0,1]$ are polynomials of degree $\le r$;*

(*ii*) $\|f-g\|_{\infty[-1,1]} \le c(r)\,\omega_r(f',1)_{\infty[-1,1]}$;

(*iii*) $\displaystyle\sum_{k=1}^{r} |g^{(k)}(0+)-g^{(k)}(0-)| \le c(r)\,\omega_r(f',1)_{\infty[-1,1]}.$

This is Theorem 3.1(i) from [2] accorded to Remark 3.

**Lemma 3** *Let f be a nondecreasing function which is the primitive of a bounded and measurable on $[-2,2]$ function $f'$ and let $g_1$ and $g_1$ be the piecewise polynomials guaranteed by Lemma 2 for the intervals $I=[-2,0]$ and $I=[0,2]$, respectively. Then*

$$\sum_{k=1}^{r} |g_2^{(k)}(0+)-g_1^{(k)}(0-)| \le c(r)\,\omega_r(f',1)_{\infty[-2,2]}.$$

This is Theorem 3.2(i) from [2] accorded to Remark 3.

The next lemma is similar to Lemma 2 and the proof runs along the lines of that of Lemma 2.

**Lemma 4** *Let f be a nondecreasing function which is the primitive of a bounded and measurable on $[-m,l]$ (m and l natural) function $f'$. For $r \ge 1$ there exists a nondecreasing continuous function $g$ on [-m,l] such that $g$ interpolates $f$ at -m, 0 and l and has the properties:*

(*i*) *The restrictions of $g$ to $[-m,0]$ and $[0,l]$ are polynomials of degree $\le r$;*

(*ii*) $\|f-g\|_{\infty[-m,l]} \le c(r)\,\omega_r(f',1)_{\infty[-\max\{m,l\},\max\{m,l\}]}$;

(*iii*) $\displaystyle\sum_{k=1}^{r} |g^{(k)}(0+)-g^{(k)}(0-)| \le c(r)\,\omega_r(f',1)_{\infty[-\min\{m,l\},-\min\{m,l\}]}.$

We use also the following fundamental Lemma of Chui, Smith and Ward (see [1] ).

**Lemma CSW.** *Let $r \ge 2$ and $d=4r^2$ and let $g$ be a nondecreasing continuous function on $[-3d,3d]$, the restriction of which to $[-3d,0]$ and to $[0,3d]$ polynomials of degree $\le r-1$. Then there is a nondecreasing spline $s$ of order $r$ and knots at the integers such that*

$$\|s-g\|_{p[-3d,3d]} = \|s-g\|_{p[-d,d]} \le c(r)\sum_{k=1}^{r-1} |g^{(k)}(0+)-g^{(k)}(0-)|.$$

## 2 Main result.

*Proof of Theorem 3.* It suffices to prove Theorem for $n > 12d$, where $d = 4r^2$. Let $F(t) = f\left(\frac{t}{n}\right)$, $t \in [0,n]$, and let $m = 2\left[\frac{n}{6d}\right]$ ( [.]-integral part ). Denote $I_1 = [0,3d]$, $I_2 = [3d,6d], ..., I_{m-1} = [3(m-2)d, 3(m-1)d]$ and $I_m = [3(m-1)d, n$. By Lemma 2 for each pair of intervals $I_{2j-1} \cup I_{2j}$, $j = 1,2,...,\frac{m}{2}-1$, there exists a monotone nondecreasing continuous function $G_j$ interpolating $F$ at $6(j-1)d$, $(6j-3)d$ and $6jd$, such that $G_j$ is a polynomial of degree $\le r-1$ on $I_{2j-1}$ and on $I_{2j}$. Also,

$$\|F - G_j\|_{\infty(I_{2j-1}\cup I_{2j})} \le c(r)\ \omega_{r-1}(F',1)_{\infty(I_{2j-1}\cup I_{2j})}.$$

and

$$\sum_{k=1}^{r-1} |G_j^{(k)}((6j-3)d+) - G_j^{(k)}((6j-3)d-)| \le c(r)\ \omega_{r-1}(F',1)_{\infty(I_{2j-1}\cup I_{2j})}.$$

Let we note that the constants in the inequalities are independent of the intervals. We must note also that the length of $I_m$ may be $> 3d$. This is the reason for the using of Lemma 4. By Lemma 4 for the last pair of intervals $I_{m-1} \cup I_m$, there exists a monotone nondecreasing continuous function $G_{\frac{m}{2}}$ interpolating $F$ at $3(m-2)d$, $3(m-1)d$ and $n$, such that $G_{\frac{m}{2}}$ is a polynomial of degree $\le r-1$ on $I_{m-1}$ and on $I_m$. Also,

$$\|F - G_{\frac{m}{2}}\|_{\infty(I_{m-1}\cup I_m)} \le c(r)\ \omega_{r-1}(F',1)_{\infty(I_{m-2}\cup I_{m-1}\cup I_m)}. \tag{3}$$

and

$$\sum_{k=1}^{r-1} |G_{\frac{m}{2}}^{(k)}(3(m-1)d+) - G_{\frac{m}{2}}^{(k)}(3(m-1)d-)| \le c(r)\ \omega_{r-1}(F',1)_{\infty(I_{m-1}\cup I_m)}. \tag{4}$$

In the right hand side of (3) and (4) we use that $3d \le n - 3(m-1)d \le 6d$ and $I_{m-2}$ exists because $m \ge 2$ $(n > 12d)$.

Now by Lemma 3, we may define a continuous nondecreasing function $G = G_j$ on $I_{2j-1} \cup I_{2j}$, $j = 1,2,...,\frac{m}{2}$ such that

$$\begin{array}{ll} \|F - G\|_{\infty(I_{2j-1}\cup I_{2j})} \le c(r)\ \omega_{r-1}(F',1)_{\infty(I_{2j-1}\cup I_{2j})}, & j < \frac{m}{2}; \\ \|F - G\|_{\infty(I_{m-1}\cup I_m)} \le c(r)\ \omega_{r-1}(F',1)_{\infty(I_{m-2}\cup I_{m-1}\cup I_m)}, & j = \frac{m}{2}. \end{array} \tag{5}$$

and for $i = 1,2,...,m-1$

$$\sum_{k=1}^{r-1} |G^{(k)}(3id+) - G^{(k)}(3id-)| \le c(r)\ \omega_{r-1}(F',1)_{\infty(I_i\cup I_{i+1})}. \tag{6}$$

Applying Lemma CSW in each pair of intervals $I_i \cup I_{i+1}$ we have a spline $S_i$ on $I_i \cup I_{i+1}$ such that $S_i = G$ outside $[(3i-1)d, (3i+1)d]$ and by (6)

$$\|S_i - G\|_{\infty[(3i-1)d,(3i+1)d]} \le c(r)\ \omega_{r-1}(F',1)_{\infty(I_i\cup I_{i+1})}. \tag{7}$$

We define the spline

$$S(t) \stackrel{\text{def}}{=} \begin{cases} S_i(t) & if\ \ t \in [(3i-1)d, (3i+1)d]\ ,\ \ i = 1,2,...,m-1; \\ G(t) & \qquad\qquad\qquad\qquad\qquad otherwise. \end{cases}$$

Now we let $s(t) = S(nt)$, $0 \le t \le 1$. Then $s \in S(r,n)$, $s$ is monotone nondecreasing and using (5), (7) and (2) we obtain

$$\begin{aligned}
&\|f-s\|^p_{p[0,1]} = \frac{1}{n}\|F-S\|^p_{p[0,n]} \\
&\le \frac{2^p}{n}\left(\|F-G\|^p_{p[0,n]} + \|G-S\|^p_{p[0,n]}\right) \\
&\le \frac{2^p}{n}\left(\sum_{j=1}^{\frac{m}{2}-1}\|F-G\|^p_{p(I_{2j-1}\cup I_{2j})} + \|F-G\|^p_{p(I_{m-1}\cup I_m)} + \sum_{i=1}^{m-1}\|G-S\|^p_{p(I_i\cup I_{i+1})}\right) \\
&\le \frac{2^p}{n}c(r)\left(\sum_{j=1}^{\frac{m}{2}-1}\int_{I_{2j-1}\cup I_{2j}}\omega^p_{r-1}(F',1)_{\infty(I_{2j-1}\cup I_{2j})}dt\right. \\
&\left.+\int_{I_{m-1}\cup I_m}\omega^p_{r-1}(F',1)_{\infty(I_{m-2}\cup I_{m-1}\cup I_m)}dt + \sum_{i=1}^{m-1}\int_{I_i\cup I_{i+1}}\omega^p_{r-1}(F',1)_{\infty(I_i\cup I_{i+1})}dt\right) \\
&\le \frac{2^p}{n}c(r)\left(\sum_{j=1}^{\frac{m}{2}-1}\int_{I_{2j-1}\cup I_{2j}}\omega^p_{r-1}(F',t;c(r))dt + \int_{I_{m-1}\cup I_m}\omega^p_{r-1}(F',t;c(r))dt\right. \\
&\left.+\sum_{i=1}^{m-1}\int_{I_i\cup I_{i+1}}\omega^p_{r-1}(F',t;c(r))dt\right) \\
&\le \frac{2^p}{n}c(r)\tau^p_{r-1}(F',c(r))_{p[0,n]} \\
&= c(r)\left(n^{-1}\tau_{r-1}(f',c(r)n^{-1})_{p[0,1]}\right)^p \\
&\le c(r)\left(c(r)n^{-1}\tau_{r-1}(f',n^{-1})_{p[0,1]}\right)^p.
\end{aligned}$$

Therefore

$$\|f-s\|_{p[0,1]} \le c(r)n^{-1}\tau_{r-1}(f',n^{-1})_{p[0,1]}.$$

# References


[1] C.K. Chui, P.W. Smith, J.D. Ward. Degree of $L_p$ approximations by monotone splines. *SIAM J. Math.Anal.* **11**, (1980) 436-447.

[2] D. Leviatan, H. N. Mhaskar. The rate of monotone spline approximation in the $L_p$-norm. *SIAM J. Math.Anal.* **13**, *5*, (1982) 866-874.

[3] Bl. Sendov and V. A. Popov. The Averaged Moduli of Smoothness. *John Willey & Sons.* (1988).



Parvan Parvanov
Department of Mathematics
Higher Transport School
1754 Slatina, Sofia, Bulgaria
e-mail pparvanov @ mail.vvtu.bg