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A Characterization of Best Multivariate Algebraic Approximations from Below and from Above in Terms of K-functionals

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This paper is the first step in the characterization of the best multivariate algebraic approximations from below and from above. Direct and inverse inequalities for the best constrained approximations in terms of appropriate K-functionals are proved.

1. Introduction

We consider measurable real-valued bounded (from below or from above) functions defined in every point of the domain $\Omega = \Pi[-1;1]$, where

$$\Pi[a;b] := \left\{x \in R^d \; ; \; x_i \in [\min\{a_i, b_i\}, \max\{a_i, b_i\}] \quad \text{for every } i = 1, \ldots, d\right\}$$

and $x = (x_1, \ldots, x_d), a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d)$ are points in R^d (d is a natural number). Here 1 and -1 mean respectively $(1, \ldots, 1)$ and $(-1, \ldots, -1)$.

Let X be a measurable subset of Ω. We shall consider the following spaces

$$L_p(X) = \left\{f; \;\; \|f\|_{p(X)} = \left\{\int_X |f(x)|^p dx\right\}^{\frac{1}{p}} < \infty\right\},$$

for $p \in [1, \infty)$ (dx means the Lebesgue measure on X) and

$$L_\infty(X) = \left\{f; \;\; \|f\|_{\infty(X)} = \sup\{|f(x)| \; ; \; x \in X\} < \infty\right\},$$

for $p = \infty$.

Here α, β, ϵ are multi-indices. If $\alpha = (\alpha_1, .., \alpha_d)$, $\alpha_s \geq 0$ for any $s = 1, ..., d$, $|\alpha| = \sum_{i=1}^{d} \alpha_i$ is the length of α. $\alpha \geq \beta$ means $\alpha_s \geq \beta_s$ for any $s = 1, ..., d$ and $\binom{\alpha}{\beta} = \Pi_{s=1}^{d} \binom{\alpha_s}{\beta_s}$.

Let r be natural. By $W_p^r(X)$ we denote the Sobolev space

$$W_p^r(X) := \left\{ f; \sum_{|\alpha|=r} \|D^\alpha f\|_{p(X)} < \infty \right\}, \quad \text{where} \quad D^\alpha = \prod_{i=1}^{d} \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}} .$$

For $v \in [-1, 1], t > 0$ we set $\psi(t, v) := t\sqrt{1 - v^2} + t^2$. For $x \in \Omega$ we denote $\Psi(t, x) := \prod_{s=1}^{d} \psi(t, x_s)$ and $\Psi^\alpha(t, x) := \prod_{s=1}^{d} \psi(t, x_s)^{\alpha_s}$. At a neighbourhood of the point $x \in \Omega$ we define by

$$U(t, x) := \{y \in \Omega \,; |x_s - y_s| \leq \psi(t, x_s) \text{ for } every \ s = 1, ..., d\}.$$

Everywhere in this paper c denotes a positive number which may depend on r, d and p. The c's may differ at each occurrence. If c depends on another parameter we indicate this using brackets.

By H_n we denote the set of all algebraic polynomials in R^d of total degree not greater than n. The best approximations by algebraic polynomials are given by

$$E(f, H_n)_{p(X)} := \inf \left\{ \|f - Q\|_{p(X)} \ ; \ Q \in H_n \right\}$$

and the best approximations from below or from above by algebraic polynomials are given respectively by

$$E^-(f, H_n)_{p(X)} := \inf \left\{ \|f - Q\|_{p(X)} \ ; \ Q \in H_n \ , Q \leq f \right\} \tag{1.1}$$

and

$$E^+(f, H_n)_{p(X)} := \inf \left\{ \|f - Q\|_{p(X)} \ ; \ Q \in H_n \ , Q \geq f \right\}, \tag{1.2}$$

whenever f is bounded from below or from above respectively.

Everywhere in this paper $l = l(r, p, d)$ is the bigger of the numbers $\left[\frac{d}{p}\right] + 1$ and r ($[\cdot]$ - integral part). We investigate the K-functionals

(1.3)
$$\begin{aligned} K^-(f, t)_p &= K^-\left(f, t; L_p, W_p^r(\Psi), W_p^l(\Psi)\right) \\ &:= \inf \left\{ \|f - g\|_{p(\Omega)} + \textstyle\sum_{|\alpha|=r,l} \|\Psi^\alpha(t) D^\alpha g\|_{p(\Omega)}; g \leq f. \ g \in W_p^l(\Omega) \right\}, \end{aligned}$$

(1.4)
$$\begin{aligned} K^+(f, t)_p &= K^+\left(f, t; L_p, W_p^r(\Psi), W_p^l(\Psi)\right) \\ &:= \inf \left\{ \|f - g\|_{p(\Omega)} + \textstyle\sum_{|\alpha|=r,l} \|\Psi^\alpha(t) D^\alpha g\|_{p(\Omega)}; g \geq f, g \in W_p^l(\Omega) \right\}, \end{aligned}$$

(1.5)

$$K\left(f,t;L_p,W_p^r(\Psi)\right) := \inf\left\{\|f-g\|_{p(\Omega)} + \sum_{|\alpha|=r}\|\Psi^\alpha(t)D^\alpha g\|_{p(\Omega)};\, g\in W_p^r(\Omega)\right\}$$

and

(1.5′)
$$K\left(f,t;L_p,W_p^r(\Psi),W_p^l(\Psi)\right) := \inf\left\{\|f-g\|_{p(\Omega)} + \textstyle\sum_{|\alpha|=r,l}\|\Psi^\alpha(t)D^\alpha g\|_{p(\Omega)} \;;\; g\in W_p^l(\Omega)\right\}.$$

Let $U\subset R^d$ be a convex body. We set

(1.6)
$$\omega_r(f,U)_p := \sup\left\{\|\Delta_{h,U}^r f(\cdot)\|_{p(U)};\;\; h\in R^d\right\},$$

where

$$\Delta_{h,U}^r f(x) := \begin{cases}\Delta_h^r f(x) & \text{if } \; x, x+rh\in U;\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\Delta_h^r f(x) = \sum_{i=0}^r (-1)^{r-i}\binom{r}{i} f(x+ih).$$

Let $\Pi=\Pi[a;b]$ and $\pi=\Pi[c;d]$ (a neighbourhood of the 0) be such that

(1.7)
$$\pi\subseteq\left(\Pi-\frac{a+b}{2}\right)\subseteq R.\pi$$

for some $R\geq 1$, where for $U\subset R^d$, $y\in R^d$ and $t>0$, we denote

$$U+y := \{x\in R^d\;;\;\; x-y\in U\},$$

$$tU := \{x\in R^d;\;\; t^{-1}x\in U\}.$$

The main result of this paper is the following statement for the best constrained approximations in terms of the K-functionals.

Theorem 1.1. *Let $1\leq p\leq\infty$, r and n be natural, $*=$ "$-$" or "$+$" and let $f\in L_p(\Omega)$ be bounded from below or from above respectively. Then we have*

(*d*)
$$E^*(f,H_{n-1})_{p(\Omega)} \leq cK^*\left(f,n^{-1};L_p,W_p^r(\Psi),W_p^l(\Psi)\right);$$

(*i*)
$$K^*\left(f,n^{-1};L_p,W_p^r(\Psi),W_p^l(\Psi)\right)\leq c\left(E^*(f,H_{n-1})_{p(\Omega)} + K\left(f,n^{-1};L_p,W_p^r(\Psi)\right)\right).$$

These inequalities are the reason for the investigations in [5], where we characterize for $r = 1$ and $r = 2$ the constrained K-functionals in terms of appropriate moduli and thus we make a characterization of the best algebraic approximations from below and from above. This theorem is proved in Section 5. Let we mention that the "classic" inverse inequality for the best constrained approximations is

$$\begin{aligned} &K^*\left(f, n^{-1}; L_p, W_p^r(\Psi), W_p^l(\Psi)\right) \\ &\le c\left(E^*(f, H_{n-1})_{p(\Omega)} + K\left(f, n^{-1}; L_p, W_p^r(\Psi), W_p^l(\Psi)\right)\right). \end{aligned}$$

We set

$$B(t,x) := \left\{y \in R^d \,;\, |y_s| \le \psi(t, x_s) \text{ for every } \; s = 1, ..., d\right\}.$$

In this paper we consider the following averaged modulus of smoothness

$$\tau_r(f, \Psi(t))_{p,p(\Omega)} := \left\{\int_\Omega \Psi(t,x)^{-1} \int_{B(t,x)} |\Delta_{v,\Omega}^r f(x)|^p dv dx\right\}^{\frac{1}{p}}, \tag{1.8}$$

which is equivalent to (1.5) and (1.5′).

Theorem 1.2. *Let $1 \le p \le \infty$, r be natural and $f \in L_p(\Omega)$. Then we have*

(r) $\qquad c\tau_r(f, \Psi(t))_{p,p(\Omega)} \le K\left(f, t, L_p, W_p^r(\Psi)\right) \le c\tau_r(f, \Psi(t))_{p,p(\Omega)};$

(r, l) $\qquad c\tau_r(f, \Psi(t))_{p,p(\Omega)} \le K\left(f, t, L_p, W_p^r(\Psi), W_p^l(\Psi)\right) \le c\tau_r(f, \Psi(t))_{p,p(\Omega)}.$

This theorem is proved in Section 4.

In this paper $meas(V)$ denotes the Lebesgue measure of the measureble set V. In order to prove Theorem 1.2 we use the following characteristic of f.

$$\tau_r(f, \pi)_{p,p(\Pi)} := \left\{\int_\Pi \frac{1}{meas(\pi)} \int_\pi |\Delta_{v,\Pi}^r f(x)|^p dv dx\right\}^{\frac{1}{p}}. \tag{1.9}$$

A relationship between (1.6) and (1.9) is

Theorem 1.3. *If (1.7) is satisfied and $f \in L_p(\Omega)$ then,*

$$c\tau_r(f, \pi)_{p,p(\Pi)} \le \omega_r(f, \Pi)_p \le cR^{d+r}\tau_r(f, \pi)_{p,p(\Pi)}.$$

This theorem is proved in Section 3.

2. Some notations and auxiliary results

Let N be a fixed natural number. We set

$$Z = \{0, 1, ..., N-1\}^d \; ; \; Z' = \{0, 1, ..., N\}^d \; ; \; \mathrm{E} = \{0, 1\}^d;$$

$$z_k = \cos(\pi - \frac{k\pi}{N}),\ k = 0, 1, ..., N,\ z_{-1} = z_0 = -1,\ \ z_{N+1} = z_N = 1.$$

For every $j = (j_1, j_2, ..., j_d) \in Z$ we denote

$$\Omega_j = [z_{j_1}, z_{j_1+1}] \times \ ... \ \times [z_{j_d}, z_{j_d+1}]$$

and for every $j \in Z'$ we denote

$$\Omega'_j = [z_{j_1-1}, z_{j_1+1}] \times \ ... \ \times [z_{j_d-1}, z_{j_d+1}].$$

We set $\mu(v) = \int_0^v e^{\frac{-1}{u-u^2}} du / \int_0^1 e^{\frac{-1}{u-u^2}} du$ for $0 < v < 1, \mu(v) = 0$ for $v \leq 0$ and $\mu(v) = 1$ for $v \geq 1$. Therefore $\mu \in C^\infty(R)$ and we define

$$\begin{aligned} &\mu_0(v) = 1 - \mu((v - z_0)/(z_1 - z_0)); \\ &\mu_s(v) = \mu((v - z_{s-1})/(z_s - z_{s-1}))(1 - \mu((v - z_s)/(z_{s+1} - z_s))) \\ &\qquad \text{for } s = 1, 2, ..., N-1; \\ &\mu_N(v) = \mu((v - z_{N-1})/(z_N - z_{N-1})). \end{aligned}$$

Finally for every $j \in Z$ we set $\mu_j(x) = \Pi_{s=1}^d \mu_{j_s}(x_s)$. Therefore for every $x \in \Omega$ we have

$$0 \leq \mu_j(x) \leq 1; \;\; \mu_j(x) = 0 \;\; if \;\; x \notin \Omega'_j; \tag{2.1}$$

$$\sum_{j \in Z'} \mu_j(x) = 1. \tag{2.2}$$

In the statements below we collect some properties of the above quantities.

Let $0 < t \leq \frac{1}{2}$ and $N = \left[\frac{2\pi}{t}\right] + 1$. Then we have

$$\Psi(t, x) \leq meas(U(t, x)) \leq 2^d \Psi(t, x); \tag{2.3}$$

$$\Psi(t, x) \sim \Psi(t, y) \;\text{ for every } y \in U(t, x); \tag{2.4}$$

$$\Psi(t, x) \sim \Psi(t, x + y) \;\text{ for every } y \in B(t, x); \tag{2.4'}$$

$$(2.5) \qquad c\Psi(t,x) \le meas(\Omega'_j) \le c\Psi(t,y) \text{ for every } y \in \Omega'_j;$$

$$(2.6) \qquad \Omega'_j \subset U(t,x) \text{ for any } x \in \Omega'_j.$$

The inequalities (2.3)-(2.6) are proved in [1]. (2.4$'$) follows from (2.3), (2.4) and definition of $B(t,x)$.

In order to prove the main result of this paper we need of the following lemmas.

Lemma 2.1. *Let $1 \le p \le \infty$, r and n be natural, $g \in W^l_p(\Omega)$, then we have*

$$E^-(g, H_{n-1})_{p(\Omega)} \le c \sum_{|\alpha|=r,l} \|\Psi^\alpha(n^{-1}) D^\alpha g\|_{p(\Omega)}.$$

This Lemma is a trivial corollary from Theorem 2 and Theorem 1 in [1], because the error of the best algebraic aprroximation from below is smaller than the error of the best one-sided algebraic aprroximation.

Lemma 2.2. *Let $1 \le p \le \infty$, r and n be natural and $f \in L_p(\Omega)$ be bounded from below. Then we have*

$$E^-(f, H_{n-1})_{p(\Omega)} \le cK^-\left(f, n^{-1}; L_p, W^r_p(\Psi), W^l_p(\Psi)\right).$$

This Lemma follows from Lemma 2.1 and Theorem 4.1 in [2] applied to the algebraic aprroximations from below.

Lemma 2.3. *Let f and g belong to $L_p(\Omega)$, then*

$$(2.7) \qquad \tau_r(f+g, \Psi(t))_{p,p(\Omega)} \le \tau_r(f, \Psi(t))_{p,p(\Omega)} + \tau_r(g, \Psi(t))_{p,p(\Omega)};$$

$$(2.8) \qquad \tau_r(f, \Psi(t))_{p,p(\Omega)} \le c\|f\|_{p(\Omega)} \text{ for } 0 < t \le \frac{1}{2};$$

$$(2.9) \qquad \tau_r(f, \Psi(t_1))_{p,p(\Omega)} \le c(A)\tau_r(f, \Psi(t_2))_{p,p(\Omega)}, \text{ if } t_1 \le t_2 \le At_1.$$

Proof. We get (2.7) and (2.9) directly from the definition (1.8). To prove (2.8) we use (2.4$'$)

$$\begin{aligned}
\tau_r(f, \Psi(t))_{p,p(\Omega)} &\le \|f\|_{p(\Omega)} + c\left\{\int_\Omega \Psi(t,x)^{-1} \int_{rB(t,x)} |f(x+y)|^p dy dx\right\}^{\frac{1}{p}} \\
&\le \|f\|_{p(\Omega)} + c\left\{\int_\Omega \int_{rB(t,x)} |f(x+y)|^p \Psi(t, x+y)^{-1} dy dx\right\}^{\frac{1}{p}} \\
&\le \|f\|_{p(\Omega)} + c\left\{\int_\Omega |f(x)|^p dx\right\}^{\frac{1}{p}} \le c\|f\|_{p(\Omega)}.
\end{aligned}$$

3. A new representation of $\omega_r(f,\Pi)_p$

Proof of Theorem 1.3. Obviously,

$$\tag{3.1} \begin{aligned}\tau_r(f,\pi)_{p,p(\Pi)} &\le c\left\{\frac{1}{meas(\pi)}\int_\pi \sup\{\int_\Pi |\Delta^r_{w,\Pi}f(x)|^p dx\ ;\ w\in R^d\ \}dw\right\}^{\frac{1}{p}}\\ &\le c\omega_r(f,\Pi)_p.\end{aligned}$$

From [6] we have that if f is defined for each $x\in R^d$, then for every $h,\ v\in R^d$ the r-th finite difference satisfies the equality

$$\tag{3.2} \Delta^r_h f(x)=\sum_{i=1}^{r}(-1)^i\binom{r}{i}\left\{\Delta^r_{\frac{i}{r}(v-h)}f(x+ih)-\Delta^r_{h+\frac{i}{r}(v-h)}f(x)\right\}.$$

For $h\in R^d$ we set $\Pi_h=\Pi\cap(\Pi-rh)$. Π_h is also parallelepiped if

$$\tag{3.3} h\in 2r^{-1}\left(\Pi-\frac{a+b}{2}\right).$$

Let (3.3) be satisfied. We devide Π_h and $r^{-1}\pi$ in 2^d parallelepipeds, respectively $B_1,\dots,B_{2^d}$ through the hyperplanes across the center, which are parallel to the coordinate hyperplanes and $\pi_1,\dots,\pi_{2^d}$ through the coordinate hyperplanes, s.t. π_i is opposite to B_i. Suppose $x\in B_s$ and $v\in r^{-1}\pi_s$ from (1.7) we have that

$$\tag{3.4} x,\ \ x+iv\ \ \text{and}\ \ x+(r-i)h+iv\ \ \text{belong to}\ \ \Pi\ \ \text{for every}\ \ i.$$

Hence for every $x\in B_s$ using (3.2) and (3.4) we get

$$\begin{aligned} meas(\pi_s)|\Delta^r_{h,\Pi}f(x)|^p &\le c\int_{\pi_s}\Big(\textstyle\sum_{i=1}^r|\Delta^r_{\frac{i}{r}(v-h),\Pi}f(x+ih)|\\ &\quad +|\Delta^r_{h+\frac{i}{r}(v-h),\Pi}f(x)|\ \Big)^p dv.\end{aligned}$$

Thus for every $x\in\Pi_h$,

$$\begin{aligned} |\Delta^r_{h,\Pi}f(x)|^p &\le c\textstyle\sum_{s=1}^{2^d}\frac{1}{meas(\pi_s)}\int_{\pi_s}\Big(\sum_{i=1}^r|\Delta^r_{\frac{i}{r}(v-h),\Pi}f(x+ih)|\\ &\quad +|\Delta^r_{h+\frac{i}{r}(v-h),\Pi}f(x)|\ \Big)^p dv.\end{aligned}$$

Taking $L_p(\Pi_h)$-norm with respect to x in the above inequality and using

$$\int_{\Pi_h}|\Delta^r_{h,\Pi}f(x)|^p dx=\int_{\Pi}|\Delta^r_{h,\Pi}f(x)|^p dx$$

in the left-hand side and x, $x + ih \in \Pi$ in the right-hand side we derive

$$(3.5)\quad \begin{aligned}\|\Delta^r_{h,\Pi} f(\cdot)\|_{p(\Pi)} \leq c\sum_{i=1}^{r} \Bigg[&\left\{ \int_\Pi \frac{1}{meas(\pi)} \int_{r^{-1}\pi} |\Delta^r_{\frac{i}{r}(v-h),\Pi} f(y)|^p dv dy \right\}^{\frac{1}{p}} \\ + &\left\{ \int_\Pi \frac{1}{meas(\pi)} \int_{r^{-1}\pi} |\Delta^r_{h+\frac{i}{r}(v-h),\Pi} f(y)|^p dv dy \right\}^{\frac{1}{p}} \Bigg].\end{aligned}$$

Let $\mathbf{v} \in r^{-1}\pi$. Then from (1.7)

$$(3.6)\qquad \frac{i}{r}(v-h) \text{ and } h + \frac{i}{r}(v-h) \text{ belong to } \frac{2R+1}{r}\pi.$$

Using (3.5) and (3.6) we have

$$(3.7)\qquad \|\Delta^r_{h,\Pi} f(\cdot)\|_{p(\Pi)} \leq c \left\{ \int_\Pi \frac{1}{meas(\pi)} \int_{\frac{2R+1}{r}\pi} |\Delta^r_{v,\Pi} f(y)|^p dv d \right\}^{\frac{1}{p}}.$$

We need of the equality

$$(3.8)\qquad \|\Delta^r_{h,\Pi} f(\cdot)\|_{p(\Pi)} \leq cR^{r+d} \left\{ \int_\Pi \frac{1}{meas(\pi)} \int_\pi |\Delta^r_{v,\Pi} f(y)|^p dv dy \right\}^{\frac{1}{p}}.$$

If $\frac{2R+1}{r} \leq 1$, then (3.8) is a trivial corollary from (3.7). Else, suppose $n = \left[\frac{2R+1}{r}\right] + 1$. From (3.7) we have

$$\|\Delta^r_{h,\Pi} f(\cdot)\|_{p(\Pi)} \leq c \left\{ \int_\Pi \frac{1}{meas(\pi)} \int_{n\pi} |\Delta^r_{v,\Pi} f(y)|^p dv dy \right\}^{\frac{1}{p}}.$$

Changing v by nw, we obtain

$$(3.9)\qquad \|\Delta^r_{h,\Pi} f(\cdot)\|_{p(\Pi)} \leq cR^d \left\{ \int_\Pi \frac{1}{meas(\pi)} \int_\pi |\Delta^r_{nw,\Pi} f(y)|^p dw dy \right\}^{\frac{1}{p}}.$$

From the definition of the r-th finite difference we have that if f is defined for each $y \in R^d$, then for every $w \in R^d$ the following equality holds

$$\Delta^r_{nw} f(y) = \sum_{k_1=0}^{n-1} \dots \sum_{k_r=0}^{n-1} \Delta^r_w f(y + k_1 w + \dots + k_r w).$$

Then

$$\Delta^r_{nw,\Pi} f(y) = \sum_{k_1=0}^{n-1} \dots \sum_{k_r=0}^{n-1} \Delta^r_{w,\Pi} f(y + k_1 w + \dots + k_r w).$$

Finally the last equality and (3.9) give (3.8).

Now from (3.8) we receive

$$\sup\left\{\|\Delta^r_{h,\Pi}f(\cdot)\|_{p(\Pi)}\ ;h\in\frac{2}{r}(\Pi-\frac{a+b}{2})\right\}\leq cR^{r+d}\tau_2^-(f,\pi)_{p,p(\Pi)}.$$

But for $h\in R^d\setminus\frac{2}{r}(\Pi-\frac{a+b}{2})$ we have $\|\Delta^r_{h,\Pi}f(\cdot)\|_{p(\Pi)}=0$ and then

$$\sup\{\|\Delta^r_{h,\Pi}f(\cdot)\|_{p(\Pi)}\ ;h\in R^d\}\leq cR^{r+d}\tau_2^-(f,\pi)_{p,p(\Pi)}.$$

This inequality and (3.1) complete the proof of Theorem 1.2. ■

Remark . Let $U\subset R^d$ be a covex neighbourhood of the 0 with a strictly positive minimal radius R_1 and with a finite maximal radius R_2. That is, $B(R_1)\subset U\subset B(R_2)$, where $B(\rho)=\{x\in R^d;\ |x|\leq\rho\}$. We can investigate

$$\tau_r(f,U)_{p,p(\Pi)}:=\left\{\int_\Pi\frac{1}{meas(U)}\int_U|\Delta^r_{v,\Pi}f(x)|^p dv dx\right\}^{\frac{1}{p}}$$

and prove in analogical way the statement which is similar to T.1.2 for the above averaged modulus.

4. Equivalence of (1.5),(1.5') and (1.8)

Here we use methods which are based on ideas of [3] and prove that the K-functionals (1.5) and (1.5′) and modulus of smoothness (1.8) are equivalent (Theorem 1.2). We start with the following

Lemma 4.1. *Let $0<t\leq\frac{1}{2}$. Then for every $f\in L_p(\Omega)$ we have*

(4.1) $$\tau_r(f,\Psi(t))_{p,p(\Omega)}\leq cK\left(f,t,L_p,W_p^r(\Psi)\right);$$

(4.2) $$K\left(f,t,L_p,W_p^r(\Psi),W_p^l(\Psi)\right)\leq c\tau_r(f,\Psi(t))_{p,p(\Omega)}.$$

Proof. Let us begin with the proof of (4.2). We set

(4.3) $$N=\left[\frac{2\pi}{t}\right]+1$$

and use the notation for Ω_j, Ω'_j and μ_j from the beginning of Section 2. We denote by $Q_j\in H_{r-1}$ the polynomial of best algebraic L_p approximation of degree $r-1$ to f in Ω'_j, $j\in Z'$. Then from the Whitney theorem and Theorem 1.3 we have

$$\|f-Q_j\|_{p(\Omega'_j)} \le c\omega_r(f,\Omega'_j)_{p(\Omega'_j)} \le c\tau_r(f,\Omega'_j)_{p,p(\Omega'_j)}. \tag{4.4}$$

We set

$$g(x)=\sum_{j\in Z'}\mu_j(x)Q_j(x). \tag{4.5}$$

From (4.3)-(4.5), (2.5) and (2.6) we obtain

$$\begin{aligned}\|f-g\|^p_{p(\Omega)} &= \|\textstyle\sum_{j\in Z'}\mu_j(f-Q_j)\|^p_{p(\Omega)}\\ &\le c\textstyle\sum_{j\in Z'}\int_{\Omega'_j}|f(x)-Q_j(x)|^p dx\\ &\le c\textstyle\sum_{j\in Z'}\tau_r(f,\Omega'_j)^p_{p,p(\Omega'_j)}\\ &= c\textstyle\sum_{j\in Z'}\int_{\Omega'_j}\frac{1}{meas(\Omega'_j)}\int_{\Omega'_j}|\Delta^r_{v,\Omega'_j}f(x)|^p dv dx\\ &\le c\textstyle\sum_{j\in Z'}\int_{\Omega'_j}\Psi(t,x)^{-1}\int_{B(t,x)}|\Delta^r_{v,\Omega}f(x)|^p dv dx\\ &= c\tau_r(f,\Psi(t))^p_{p,p(\Omega)}\end{aligned} \tag{4.6}$$

Fix α , $|\alpha|=r$ or $|\alpha|=l$. Let $x\in\Omega_j$, $j\in Z$. From the definitions of $\mu(x)$, $Q_j(x)$ and $g(x)$ we have

$$g(x)=Q_j(x)+\sum_{\epsilon\in E}\mu_{j+\epsilon}(x)\left(Q_{j+\epsilon}(x)-Q_j(x)\right)$$

and then from the last equality and $D^\alpha Q_j=0$, it follows that

$$D^\alpha g(x)=\sum_{\epsilon\in E}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}\mu_{j+\epsilon}(x)D^\beta\left(Q_{j+\epsilon}(x)-Q_j(x)\right)$$

Now using (2.5), (2.6), the definitions of μ_j and Q_j, Markov's inequality and (4.4) we have

$$\begin{aligned}&\|\Psi^\alpha(t)D^\alpha g\|_{p(\Omega_j)}\le c\Psi^\alpha(t,z_j)\|D^\alpha g\|_{p(\Omega_j)}\\ &\le c\Psi^\alpha(t,z_j)\textstyle\sum_{\epsilon\in E}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\|D^{\alpha-\beta}\mu_{j+\epsilon}\|_{\infty(\Omega_j)}\|D^\beta\left(Q_{j+\epsilon}-Q_j\right)\|_{p(\Omega_j)}\\ &\le c\Psi^\alpha(t,z_j)\textstyle\sum_{\epsilon\in E}\sum_{0\le\beta\le\alpha}\left(\Psi^{\alpha-\beta}(t,z_j)\right)^{-1}\left(\Psi^\beta(t,z_j)\right)^{-1}\|Q_{j+\epsilon}-Q_j\|_{p(\Omega_j)}\\ &\le c\textstyle\sum_{\epsilon\in E}\|Q_{j+\epsilon}-Q_j\|_{p(\Omega_j)}\le c\sum_{\epsilon\in E}\left(\|f-Q_{j+\epsilon}\|_{p(\Omega_j)}+\|f-Q_j\|_{p(\Omega_j)}\right)\le\\ &\le c\tau_r(f,\Omega'_j)_{p,p(\Omega'_j)}.\end{aligned}$$

Hence,

$$\begin{aligned}\|\Psi^\alpha(t)D^\alpha g\|^p_{p(\Omega)}&\le c\textstyle\sum_{j\in Z'}\tau_r(f,\Omega'_j)^p_{p,p(\Omega'_j)}\\ &= c\textstyle\sum_{j\in Z'}\int_{\Omega'_j}\frac{1}{meas(\Omega'_j)}\int_{\Omega'_j}|\Delta^r_{v,\Omega'_j}f(x)|^p dv dx\\ &\le c\textstyle\sum_{j\in Z'}\int_{\Omega'_j}\Psi(t,x)^{-1}\int_{B(t,x)}|\Delta^r_{v,\Omega}f(x)|^p dv dx\\ &= c\tau_r(f,\Psi(t))^p_{p,p(\Omega)}.\end{aligned} \tag{4.7}$$

In this way (4.2) follows from (4.6), (4.7) and (1.5).

We turn our attention to (4.1). Let $\alpha = (\alpha_1, ..., \alpha_d)$, $|\alpha| = r$ be multi-index and $z = (z_1, ..., z_d) \in R^d$. We define

$$|z^\alpha| = \Pi_{i=1}^d |z_i|^{\alpha_i}$$

Immediately from the definition of the finite difference (see also [3]) we have

$$\Delta_{z,\Omega}^r f(x) = \int_0^z ... \int_0^z f_z^{(r)}(x + y_1 + ... + y_r) dy_1 ... dy_r$$

for f in $W_p^r(\Omega)$, where we assume $f_z^{(r)}(y) = 0$ when y does not belong to Ω.

From the other hand,

$$f_z^{(r)}(y) = \sum_{|\alpha|=r} \binom{r}{\alpha} D^\alpha f(y) |z^\alpha| \, |z|^{-r}.$$

Then using the last equality, after a change of variables and Hölder's inequality we have

$$\begin{aligned} |\Delta_{z,\Omega}^r f(x)| &\le \textstyle\sum_{|\alpha|=r} |z^\alpha| \, |z|^{-r} \binom{r}{\alpha} \int_0^z ... \int_0^z |D^\alpha f(x + y_1 + ... + y_r)| dy_1 ... dy_r \\ &= \textstyle\sum_{|\alpha|=r} |z^\alpha| \, |z|^{-1} \binom{r}{\alpha} \int_0^{rz} |D^\alpha f(x + y)| dy \\ &\le c \textstyle\sum_{|\alpha|=r} |z^\alpha| \, |z|^{-1} |rz|^{1-\frac{1}{p}} \{\int_0^{rz} |D^\alpha f(x + y)|^p dy\}^{\frac{1}{p}} \\ &= c \textstyle\sum_{|\alpha|=r} |z^\alpha| \, |z|^{-\frac{1}{p}} \{\int_0^{rz} |D^\alpha f(x + y)|^p dy\}^{\frac{1}{p}}. \end{aligned}$$

Here we assume $D^\alpha f(y) = 0$ when y does not belong to Ω. Then,

$$\begin{aligned} &\left\{\Psi(t, x)^{-1} \textstyle\int_{B(t,x)} |\Delta_{z,\Omega}^r f(x)|^p dz\right\}^{\frac{1}{p}} \\ &\le c \textstyle\sum_{|\alpha|=r} \left\{\Psi(t, x)^{-1} \int_{B(t,x)} |z^\alpha|^p |z|^{-1} \int_0^{rz} |D^\alpha f(x + y)|^p dy dz\right\}^{\frac{1}{p}} \\ &\le c \textstyle\sum_{|\alpha|=r} \left\{\Psi(t, x)^{-1} \int_{rB(t,x)} |D^\alpha f(x + y)|^p dy \int_{B(t,x)\setminus\frac{1}{r}B(t,x)} |z^\alpha|^p |z|^{-1} dz\right\}^{\frac{1}{p}} \\ &\le c \textstyle\sum_{|\alpha|=r} \left\{\Psi(t, x)^{-1} \Psi(t, x)^\alpha \int_{rB(t,x)} |D^\alpha f(x + y)|^p dy\right\}^{\frac{1}{p}}. \end{aligned}$$

From the above inequality, (1.8) and (2.4′) we obtain

$$\begin{aligned} \tau_r(f, \Psi(t))_{p,p(\Omega)} &\le c \left\{\textstyle\int_\Omega \Psi(t, x)^{-1} \Psi(t, x)^\alpha \int_{rB(t,x)} |D^\alpha f(x + y)|^p dy dx\right\}^{\frac{1}{p}} \\ &\le c \left\{\textstyle\int_\Omega \Psi(t, x)^\alpha \int_{rB(t,x)} |D^\alpha f(x + y)|^p \Psi(t, x + y)^{-1} dy dx\right\}^{\frac{1}{p}} \quad (4.8) \\ &\le c \{\textstyle\int_\Omega \Psi(t, x)^\alpha |D^\alpha f(x)|^p dx\}^{\frac{1}{p}} \le c \sum_{|\alpha|=r} \|\Psi^\alpha(t) D^\alpha f\|_{p(\Omega)}. \end{aligned}$$

Finally we prove (4.1). Let g be an arbitrary function in $W_p^r(\Omega)$. Using Lemma 2.3 (inequalities (2.7) and (2.9)) and (4.8) we have

$$\begin{aligned} \tau_r(f, \Psi(t))_{p,p(\Omega)} &= \tau_r((f - g) + g, \Psi(t))_{p,p(\Omega)} \\ &\le c\tau_r((f - g), \Psi(t))_{p,p(\Omega)} + \tau_r(g, \Psi(t))_{p,p(\Omega)} \\ &\le c \left(\|f - g\|_{p(\Omega)} + \textstyle\sum_{|\alpha|=r} \|\Psi^\alpha(t) D^\alpha g\|_{p(\Omega)}\right). \end{aligned}$$

Taking an infimum on all $g \in W_p^r(\Omega)$ in the above inequality, we prove (4.1).

Proof of Theorem 1.2.

We have to investigate only the case $t > \frac{1}{2}$, because for $0 < t \leq \frac{1}{2}$ Theorem 1.2 is equal to Lemma 4.1. From Theorem 1.3, Lemma 4.1 (4.1) with $t = \frac{1}{2}$ and the monotonicity of the K-functional (1.5) with respect to t we get

$$(4.9)\qquad \begin{aligned} \tau_r(f,\Psi(t))_{p,p(\Omega)} &\leq c\omega_r(f,\Omega)_p \leq c\tau_r(f,\Psi(2^{-1}))_{p,p(\Omega)} \\ &\leq cK\left(f,\tfrac{1}{2};L_p,W_p^r(\Psi)\right) \leq cK\left(f,t;L_p,W_p^r(\Psi)\right), \end{aligned}$$

which proves the left inequality of Theorem 1.2(r).

From [4] we have that there are $R \in H_{r-1}$, such that

$$\|f-R\|_{p(\Omega)} \leq c\omega_r(f,\Omega)_p.$$

Then from (1.5), Theorem 1.3 and Lemma 2.3(2.9) we obtain

$$(4.10)\qquad \begin{aligned} K\left(f,t;L_p,W_p^r(\Psi),W_p^l(\Psi)\right) &\leq c\|f-R\|_{p(\Omega)} \leq c\omega_r(f,\Omega)_p \\ &\leq c\tau_r(f,\Psi(2^{-1}))_{p,p(\Omega)} \leq c\tau_r(f,\Psi(t))_{p,p(\Omega)}. \end{aligned}$$

This is the right inequality of Theorem 1.2(r,l). Then (4.9), (4.10) and the trivial inequality

$$K\left(f,t,L_p,W_p^r(\Psi)\right) \leq K\left(f,t,L_p,W_p^r(\Psi),W_p^l(\Psi)\right)$$

give Theorem 1.2.

5. Main result

Proof of Theorem 1.1.

We first prove Theorem 1.1 for the approximation from below, i.e. in the case $* =$ " $-$ ".

Inequality (d) follows immediately from Lemma 2.2. From Theorem 1.2 we have

$$(5.1)\qquad K\left(f,t;L_p,W_p^r(\Psi),W_p^l(\Psi)\right) \leq cK\left(f,t;L_p,W_p^r(\Psi)\right).$$

(5.1) and Theorem 4.2 from [2] prove Theorem 1.1 for the approximation from below.

Using that $E^+(f) = E^-(-f)$, $K^+(f) = K^-(-f)$ (with one and the same values of the parameters) we complete the proof of Theorem 1.1.

□

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