Weighted Approximation by Baskakov-Type Operators

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Abstract

The uniform weighted approximation errors of Baskakov-type operators are characterized for weights of the form $\left(\frac{x}{1+x}\right)^{\gamma_0}(1+x)^{\gamma_\infty}$ for $\gamma_0, \gamma_\infty \in [-1, 0]$. Direct and strong converse theorems are proved in terms of the weighted K-functional.

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1 Introduction

Baskakov [2] introduced the linear positive operators

$$B_n f(x) = \sum_{k=0}^{\infty} P_{n,k}(x) f\left(\frac{k}{n}\right)$$

for approximation of bounded and continuous on $[0, \infty)$ functions f, where $P_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$, $k \in \mathbb{N}_0 = \mathbb{N} + \{0\}$, denote the Baskakov basic functions. Following the Durrmeyer modification [5, 3] of Bernstein polynomials, Sahai and Prasad [14] modified the operators B_n for integrable on $[0, \infty)$ function f as follows

$$\tilde{B}_n f(x) = \sum_{k=0}^{\infty} P_{n,k}(x)(n-1) \int_0^{\infty} P_{n,k}(y) f(y) dy.$$

Another modification was introduced by Agrawal and Thamer [1]

$$\bar{B}_n f(x) = P_{n,0}(x) f(0) + \sum_{k=1}^{\infty} P_{n,k}(x)(n-1) \int_0^{\infty} P_{n,k-1}(y) f(y) dy.$$

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Although similar to the Goodman and Sharma [8] modification of Bernstein polynomials, operators $\bar{B}_n$ lack some of its nice features. Both $\tilde{B}_n$ and $\bar{B}_n$ do not preserve all linear functions and do not commute with the weighted second derivative operator.

The Baskakov-type operators discussed in this paper are given for natural n by

$$\begin{gathered} V_n f(x) = \sum_{k=0}^{\infty} P_{n,k}(x) v_{n,k}(f) \\ v_{n,0}(f) = f(0); \quad v_{n,k}(f) = (n+1) \int_0^{\infty} P_{n+2,k-1}(y) f(y)\, dy, \quad k \in \mathbb{N}, \end{gathered} \tag{1.1}$$

where f is Lebesgue measurable on $(0,\infty)$ with a finite limit $f(0)$ at 0. As far as we know the operators V_n were introduced in 2005 by Finta [6]. He established a strong converse theorem of type B (in the terminology of [4]) for V_n. The study of operators (1.1) is continued in [7, 9, 10]. In the present article we show that operators V_n are related to Baskakov operators in the same way as Goodman–Sharma operators are related to Bernstein polynomials.

We start with some notations. The first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^2 g(x) = g''(x)$. By $\psi(x) = x(1+x)$ we denote the weight which is naturally connected with the second derivatives of the Baskakov-type operators (1.1). Our main goal in this paper is the characterization of the uniform weighted approximation error $\|w(f - V_n f)\|$ of operators (1.1) for weight functions given by

$$w(x) = w(\gamma_0, \gamma_\infty; x) = \left(\frac{x}{1+x}\right)^{\gamma_0} (1+x)^{\gamma_\infty} \tag{1.2}$$

for $x \in [0,\infty)$ and real γ_0, γ_∞. The result of Theorem 1.1 below is valid for values of the powers γ_0, γ_∞ in the range $[-1, 0]$, while some other statements are formulated for arbitrary $\gamma_\infty \le 0$.

Let us emphasize that, given the sequence of operators V_n, we investigate which is the variety of weights $w(\gamma_0, \gamma_\infty)$ allowing a characterization of the approximation error $\|w(f - V_n f)\|$. On the other hand, for a given weight w we are not interested in modifying Baskakov operators to some $\tilde{V}_n$ in order to ensure convergence of $\|w(f - \tilde{V}_n f)\|$ to 0.

By $C[0,\infty)$ we denote the space of all continuous on $[0,\infty)$ functions. The functions from $C[0,\infty)$ are not expected to be bounded or uniformly continuous. By $L_\infty[0,\infty)$ we denote the space of all Lebesgue measurable and essentially bounded in $[0,\infty)$ functions equipped with the uniform norm $\|\cdot\|$. For a weight function w we set $C(w) = \{f \in C[0,\infty) : wf \in L_\infty[0,\infty)\}$ and

$$W^2(w\psi) = \left\{g, g' \in AC_{loc}(0,\infty) : w\psi D^2 g \in L_\infty[0,\infty)\right\},$$

where $AC_{loc}(0,\infty)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,\infty)$. As Lemma 2.3 shows every function from $W^2(w(0,\gamma_\infty)\psi)$ can be defined at 0 in a way to be continuous.

Set $C_0(w) = \{f \in C(w) : f(0) = 0\}$. Similarly, by $W_0^2(w\psi)$ we denote the subspace of $W^2(w\psi)$ of functions g satisfying the additional boundary condition

$$\lim_{x\to 0+0} \psi(x)D^2g(x) = 0. \tag{1.3}$$

Note that the boundary conditions for both $C_0(w)$ and $W_0^2(w\psi)$ do not depend on the weight w. These conditions are essential when the weight w does not go to ∞ at 0, while for $\gamma_0 < 0$ we have $C_0(w) = C(w)$ and $W_0^2(w\psi) = W^2(w\psi)$.

The weighted approximation error of V_n will be compared with the K-functional between the weighted spaces $C(w)$ and $W^2(w\psi)$, which for every

$$f \in C(w) + W^2(w\psi) = \{f_1 + f_2 : f_1 \in C(w), f_2 \in W^2(w\psi)\}$$

and $t > 0$ is defined by

$$K_w(f,t) = \inf\left\{\|w(f-g)\| + t\|w\psi D^2g\| : g \in W^2(w\psi), f-g \in C(w)\right\}. \tag{1.4}$$

The above formula is a standard definition of K-functional in interpolation theory. In approximation theory the condition $f - g \in C(w)$ in (1.4) is usually omitted because in the predominant number of cases the second interpolation space is embedded in the first one. However, the study of Baskakov-type operators (1.1) involves interpolation between $C(w)$ and $W^2(w\psi)$, as $W^2(w\psi)\backslash C(w)$ is of infinite dimension for some of the weights w that satisfies the assumptions of Theorem 1.1, i.e. $w(x) = w(\gamma_0, \gamma_\infty)$ with $\gamma_0, \gamma_\infty \in [-1, 0]$.

If $-1 < \gamma_\infty < 0$ and $\gamma_0 \in [-1, 0]$ then $C(w)+W^2(w\psi) = C(w)+\pi_1$, where π_1 is the set of all algebraical polynomials of degree 1. Note that π_1 is the null space of the operator D^2. But for $\gamma_\infty = 0$ or for $\gamma_\infty = -1$ the space $C(w)+W^2(w\psi)$ is essentially bigger than $C(w) + \pi_1$. Thus, if $f(x) = x/e$ for $x \in [0, e]$ and $f(x) = \log x$ for $x \in [e,\infty)$, then $f \in (C(w) + W^2(w\psi))\backslash(C(w) + \pi_1)$ for $\gamma_\infty = 0$ and $\gamma_0 \in [-1, 0]$. Also, if $f(x) = x^2/e$ for $x \in [0, e]$ and $f(x) = x \log x$ for $x \in [e,\infty)$, then $f \in (C(w) + W^2(w\psi))\backslash(C(w) + \pi_1)$ for $\gamma_\infty = -1$ and $\gamma_0 \in [-1, 0]$. The non-emptiness of $(C(w) + W^2(w\psi))\backslash(C(w) + \pi_1)$ is determined by the increase of $\psi(x)$ as x^2 at infinity. In this respect operators (1.1) behave differently than Goodman–Sharma operators (see [12]), where always $C(w) + W^2(w\psi) = C(w) + \pi_1$ (with proper weights for the finite interval).

Before stating our main results we sketch some properties of operators V_n. They preserve the linear functions, which is an advantage when compared with $\tilde{B}_n$ or $\bar{B}_n$. In the present paper we establish several new properties of V_n, which are analogous with those of Goodman–Sharma operators. For example, Theorem 2.6 shows that V_n and V_m commutes for all indexes n and m, in Theorem 2.5 we prove that V_n commutes with ψD^2 and Lemma 2.2 shows that D^2V_nf depends only on the difference $V_nf - V_{n+1}f$. The last two properties essentially simplify the proof of Theorem 1.3.

Our main result is the following theorem, consisting of a direct inequality (1.6) and a strong converse inequality (1.7) of type A in the terminology of [4].

Theorem 1.1. *Let $w = w(\gamma_0, \gamma_1)$ be given by* (1.2) *with $\gamma_0, \gamma_1 \in [-1, 0]$. Then for every $f \in C(w) + W^2(w\psi)$ and every $n \in \mathbb{N}$, $n \geq 4$, we have*

$$\|w(f - V_n f)\| \leq 2K_w \left(f, \frac{1}{2n}\right) \leq 13.7\|w(f - V_n f)\|. \tag{1.5}$$

Taking into account that $w(\gamma_0, \gamma_1)^{-1}$ is concave if and only if $\gamma_0, \gamma_1 \in [-1, 0]$ we see that the first inequality in (1.5) is contained in the following direct theorem.

Theorem 1.2. *Let w^{-1} be concave. Then for every $f \in C(w) + W^2(w\psi)$ and for every $n \in \mathbb{N}$ we have*

$$\|w(V_n f - f)\| \leq 2K_w \left(f, \frac{1}{2n}\right). \tag{1.6}$$

The second inequality in (1.5) is a consequence of the following strong inverse theorem of type A.

Theorem 1.3. *Let $w = w(\gamma_0, \gamma_1)$ be given by* (1.2) *with $\gamma_0, \gamma_\infty \in [-1, 0]$. Then for every $f \in C(w) + W^2(w\psi)$ and for every $n \in \mathbb{N}$, $n \geq 4$, we have*

$$K_w \left(f, \frac{1}{2n}\right) \leq \frac{575}{84}\|w(f - V_n f)\|. \tag{1.7}$$

Theorem 1.2 extends or improves several results from [7, 9, 10]. Theorem 1.3 improves the result in [6] in the following directions: (i) (1.7) is a stronger inequality with only one term in the right-hand side; (ii) the inverse theorem holds for wide range of weights $w(\gamma_0, \gamma_1)$ instead of only for $w = 1$; (iii) the statement holds for $f \in C(w) + W^2(w\psi)$ instead of for $f \in C(w)$; (iv) (1.7) comes with the explicit small constant 575/84 in the right-hand side.

As seen from Theorem 1.1 operators (1.1) are saturated with saturation order n^{-1} and $W^2(w\psi)$ is contained in the saturation class (both classes actually coincide). But Theorem 1.1 *does not imply* for all $f \in C(w) + W^2(w\psi)$ that $\|w(f - V_n f)\| \to 0$ or $K_w\left(f, (2n)^{-1}\right) \to 0$ when $n \to \infty$. In fact, both quantities in (1.5) do not tend to zero with $n \to \infty$ for some functions in $C(w)$. In order to ensure convergence to zero of these quantities one may need to impose additional restrictions on the behavior of f at 0 and at ∞. At 0 these restrictions are the same as for the Goodman–Sharma operators, namely, $\lim_{x\to 0+} x^{\gamma_0} f(x) = 0$ for $-1 < \gamma_0 < 0$ or the existence of $\lim_{x\to 0+} x^{-1} f(x)$ for $\gamma_0 = -1$. In the same time, at ∞ function f should not vary very fast in order to allow approximation in $C(w)$ with functions from $W^2(w\psi)$.

Although Theorems 1.1, 1.2 and 1.3 are stated for integer n they also hold true if n is assumed to be a continuous positive parameter. In this case

$$P_{n,k}(x) = \frac{\Gamma(n+k)}{k!\Gamma(n)} x^k (1+x)^{-n-k},$$

where Γ stands for the Gamma function. V_n is defined again by (1.1). We shall use the above definition of $P_{n,k}$ only in the proof of Lemma 2.1, while for the remaining part of the paper the index of V_n will be assumed integer.

The paper is organized as follows. Some auxiliary results about operators V_n are proved in Section 2. Theorem 1.2 is proved in Section 3, while Theorem 1.3 is proved in Section 4.

2 Auxiliary results

We first observe that the operators V_n given by (1.1) are well defined for big n's on functions f with a polynomial growth at infinity. More precisely, if $w(0,\gamma_\infty)f \in L_\infty[0,\infty)$ for some (negative) γ_∞, then $V_n f$ is defined for every natural $n > -\gamma_\infty - 1$. The boundedness of V_n in this case is given in

Lemma 2.1. *Let $\gamma_\infty \le 0$ and $n \in \mathbb{N}$, $n > -\gamma_\infty - 1$. If $f \in C(w)$ with $w = w(0,\gamma_\infty)$, then $\|wV_n f\| \le c\|wf\|$ with c depending only on γ_∞.*

Proof. From (1.1) for $k \in \mathbb{N}$ we get ($\gamma = \gamma_\infty$)

$$v_{n,k}(w(0,-\gamma)) = \frac{\Gamma(n+\gamma+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+\gamma+1+k)}. \tag{2.1}$$

Representation (2.1) is trivially true for $k = 0$ too. For every $\alpha \in \mathbb{R}$ we have $\Gamma(y+\alpha)/\Gamma(y) = O(y^\alpha)$ for $y \to \infty$, which implies the existence of a constant c depending only on γ such that

$$\frac{\Gamma(n-\gamma)\Gamma(n+k)}{\Gamma(n)\Gamma(n+k-\gamma)}\frac{\Gamma(n+\gamma+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+\gamma+1+k)} \le c$$

for every $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, $n > -\gamma - 1$. Now from (1.1), (2.1) and the above inequality we get

$$\begin{aligned}
&|w(x)V_n f(x)| \le w(x)\sum_{k=0}^{\infty} P_{n,k}(x)v_{n,k}(w(0,-\gamma))\|w(0,\gamma)f\| \\
&= \sum_{k=0}^{\infty} P_{n-\gamma,k}(x)\frac{\Gamma(n-\gamma)\Gamma(n+k)}{\Gamma(n)\Gamma(n+k-\gamma)}\frac{\Gamma(n+\gamma+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+\gamma+1+k)}\|wf\| \le c\|wf\|,
\end{aligned}$$

which proves the lemma. $\square$

Throughout the article we shall employ the convention $P_{m,j}(x) \equiv 0$ for $j = -1,-2,\dots$. Next, we give some identities involving the Baskakov basic functions, which follows by their definition. For $j \in \mathbb{N}_0$, $n \in \mathbb{N}$ we have

$$mxP_{m+1,j-1}(x) = jP_{m,j}(x), \tag{2.2}$$

$$m(1+x)P_{m+1,j}(x) = (m+j)P_{m,j}(x), \tag{2.3}$$

$$m(m+1)\psi(x)P_{m+2,j}(x) = (j+1)(m+j+1)P_{m,j+1}(x), \tag{2.4}$$

$$DP_{m,j}(x) = \frac{j-mx}{\psi(x)}P_{m,j}(x), \tag{2.5}$$

$$D^2P_{m,j}(x) = \left[\left(\frac{j-mx}{\psi(x)}\right)^2 - \frac{m\psi(x)+(j-mx)(1+2x)}{\psi^2(x)}\right]P_{m,j}(x), \tag{2.6}$$

$$DP_{m,j}(x) = m[P_{m+1,j-1}(x) - P_{m+1,j}(x)], \tag{2.7}$$

$$D^2P_{m,j}(x) = m(m+1)[P_{m+2,j-2}(x) - 2P_{m+2,j-1}(x) + P_{m+2,j}(x)]. \tag{2.8}$$

Next, we collect from [2] and [11] some basic results for the original Baskakov operators. With the notation $e_j(x) = x^j,\ j = 0,1,2$ we have

$$B_n \text{ is a linear, positive operator;} \tag{2.9}$$

$$B_ne_0(x) = e_0(x),\ B_ne_1(x) = e_1(x); \tag{2.10}$$

$$B_ne_2(x) = e_2(x) + \frac{1}{n}\psi(x); \tag{2.11}$$

$$\|B_nf\| \le \|f\| \text{ for } f \in C(1); \tag{2.12}$$

Using (1.1), (2.10), (2.11), (2.12) and the basic normalization equality

$$(n-1)\int_0^\infty P_{n,k}(t)\,dt = 1 \quad \text{for } k,n \in \mathbb{N},\ n \ge 2, \tag{2.13}$$

we get the following basic properties of the operators V_n:

$$V_n \text{ is a linear, positive operator;} \tag{2.14}$$

$$V_ne_0(x) = e_0(x),\ V_ne_1(x) = e_1(x); \tag{2.15}$$

$$V_ne_2(x) = e_2(x) + \frac{2}{n-1}\psi(x), \quad n \ge 2; \tag{2.16}$$

$$\|V_nf\| \le \|f\| \text{ for } f \in C(1). \tag{2.17}$$

Using (2.14) and (2.15) we obtain

$$V_nf \le f \text{ for every concave function } f. \tag{2.18}$$

One important property of the sequence V_n is

Lemma 2.2. *If $f \in C(w(0,\gamma_\infty))$ and $n \in \mathbb{N}$, $n > -\gamma_\infty - 1$, then*

$$V_nf(x) - V_{n+1}f(x) = \frac{1}{n(n+1)}\psi(x)D^2(V_nf)(x).$$

Proof. We write $V_n f(x) = A_n f(x) + S_n f(x)$, where

$$A_n f(x) = P_{n,0}(x) f(0), \quad S_n f(x) = \sum_{k=1}^{\infty} P_{n,k}(x) v_{n,k}(f).$$

From the identities (2.2), (2.3) we get

$$(n+1) v_{n+1,k}(f) = (n+k+1) v_{n,k}(f) - k v_{n,k+1}(f). \tag{2.19}$$

Now from (2.19) and (2.2) we obtain

$$\begin{aligned} S_{n+1} f(x) &= \sum_{k=1}^{\infty} P_{n+1,k}(x) \left(\frac{n+k+1}{n+1} v_{n,k}(f) - \frac{k}{n+1} v_{n,k+1}(f) \right) \\ &= \sum_{k=1}^{\infty} P_{n+1,k}(x) \frac{n+k+1}{n+1} v_{n,k}(f) - \sum_{k=2}^{\infty} P_{n+1,k-1}(x) \frac{k-1}{n+1} v_{n,k}(f) \\ &= \sum_{k=1}^{\infty} P_{n,k}(x) v_{n,k}(f) \left(\frac{(n+k)(n+k+1)}{n(n+1)} \frac{1}{1+x} - \frac{k(k-1)}{n(n+1)} \frac{1}{x} \right). \end{aligned}$$

Using the above representation and (2.5), (2.6) we obtain

$$\begin{aligned} & S_n f(x) - S_{n+1} f(x) \\ &= \sum_{k=1}^{\infty} P_{n,k}(x) v_{n,k}(f) \frac{n(n+1)\psi(x) + k(k-1)(1+x) - (n+k)(n+k+1)x}{n(n+1)\psi(x)} \\ &= \frac{\psi(x)}{n(n+1)} \sum_{k=1}^{\infty} P_{n,k}(x) v_{n,k}(f) \left[\left(\frac{k-nx}{\psi(x)} \right)^2 - \frac{n\psi(x) + (k-nx)(1+2x)}{\psi^2(x)} \right] \\ &= \frac{\psi(x)}{n(n+1)} \sum_{k=1}^{\infty} D^2 P_{n,k}(x) v_{n,k}(f) = \frac{\psi(x)}{n(n+1)} D^2(S_n f)(x). \end{aligned} \tag{2.20}$$

For the other part of the difference $V_n f(x) - V_{n+1} f(x)$ we have

$$\begin{aligned} & A_n f(x) - A_{n+1} f(x) \\ &= \left((1+x)^{-n} - (1+x)^{-n-1} \right) f(0) = \frac{\psi(x)}{n(n+1)} D^2(A_n f)(x). \end{aligned} \tag{2.21}$$

Finally, (2.20) and (2.21) prove the lemma. ∎

The following lemma contains some boundary properties of the functions in $W^2(w(0,\gamma_\infty)\psi)$.

Lemma 2.3. *If $g \in W^2(w(0,\gamma_\infty)\psi)$ and $n \in \mathbb{N}$, $n > -\gamma_\infty - 1$, then*

$$\lim_{x\to 0} x D g(x) = 0, \quad \lim_{x\to\infty} (1+x)^{-n} D g(x) = 0, \quad \lim_{x\to\infty} (1+x)^{-n-1} g(x) = 0. \tag{2.22}$$

Moreover, g has a finite limit at 0.

Proof. The first limit follows from $g'(x) = g'(1) + \int_1^x g''(t)\,dt$ when the integral is evaluated by $|\log x| \|w(0,\gamma_\infty)\psi g''\|$ for $0 < x \le 1$. For the second limit we use the same formula with the bound $\int_1^x t^{-1}(1+t)^{-\gamma_\infty - 1}\,dt \|w(0,\gamma_\infty)\psi g''\|$ for $x \ge 1$. Similarly, we obtain the last limit in (2.22) using

$$g(x) = g(1) + g'(1)(x-1) + \int_1^x (x-t)g''(t)\,dt. \tag{2.23}$$

Representation (2.23) also gives $g(0+0) = g(1) - g'(1) + \int_0^1 tg''(t)\,dt$, which completes the proof. $\square$

Now, we apply Lemma 2.3 in the proof of

Lemma 2.4. *If $g \in W^2(w(0,\gamma_\infty)\psi)$ and $n \in \mathbb{N}$, $n > -\gamma_\infty - 1$, then*

$$\psi(x)D^2(V_n g)(x) = \sum_{k=1}^{\infty} v_{n,k}(\psi D^2 g) P_{n,k}(x).$$

Proof. Using (2.7) we get for the first derivative of $V_n g$ the representation

$$\begin{aligned} D(V_n g)(x) &= n\sum_{k=1}^{\infty} v_{n,k}(g)P_{n+1,k-1}(x) - n\sum_{k=0}^{\infty} v_{n,k}(g)P_{n+1,k}(x) \\ &= n\sum_{k=0}^{\infty}[v_{n,k+1}(g) - v_{n,k}(g)]P_{n+1,k}(x) \end{aligned}$$

From the above representation we obtain for the second derivative

$$D^2(V_n g)(x) = n(n+1)\sum_{k=0}^{\infty}[v_{n,k+2}(g) - 2v_{n,k+1}(g) + v_{n,k}(g)]P_{n+2,k}(x). \tag{2.24}$$

Now from (2.4) and (2.24) we get

$$\begin{aligned} &\psi(x)D^2(V_n g)(x) \\ &\quad = \sum_{k=0}^{\infty}[v_{n,k+2}(g) - 2v_{n,k+1}(g) + v_{n,k}(g)](k+1)(n+k+1)P_{n,k+1}(x) \\ &\quad = \sum_{k=1}^{\infty}[v_{n,k+1}(g) - 2v_{n,k}(g) + v_{n,k-1}(g)]k(n+k)P_{n,k}(x). \end{aligned} \tag{2.25}$$

For the evaluation of $v_{n,k}(\psi D^2 g)$ we first apply (2.4) followed by twice differentiation by part together with Lemma 2.3 and finally we use (2.8) to get for every $k \in \mathbb{N}$

$$
\begin{aligned}
&v_{n,k}(\psi D^2 g) \\
&\quad = \int_0^\infty (n+1)P_{n+2,k-1}(t)\psi(t)D^2 g(t)dt = \frac{k(n+k)}{n}\int_0^\infty P_{n,k}(t)D^2 g(t)dt \\
&\quad = -\frac{k(n+k)}{n}\int_0^\infty DP_{n,k}(t)Dg(t)dt = \frac{k(n+k)}{n}\int_0^\infty D^2 P_{n,k}(t)g(t)dt \\
&\quad = k(n+k)(n+1)\int_0^\infty \left[P_{n+2,k-2}(t) - 2P_{n+2,k-1}(t) + P_{n+2,k}(t)\right] g(t)dt \\
&\quad = k(n+k)\left[v_{n,k-1}(g) - 2v_{n,k}(g) + v_{n,k+1}(g)\right]. \qquad (2.26)
\end{aligned}
$$

The above proof of (2.26) is valid for $k \geq 2$. The final formula is also correct for $k = 1$ but one has to take into account the additional term with $v_{n,0}(g) = g(0)$ produced by the second integration by parts.

Finally, (2.25) and (2.26) prove the lemma. □

From Lemma 2.4 and boundary condition (1.3) we immediately get

Theorem 2.5. *If $g \in W_0^2(w(0,\gamma_\infty)\psi)$ and $n \in \mathbb{N}$, $n > -\gamma_\infty - 1$, then*

$$\psi(x)D^2(V_n g)(x) = V_n(\psi D^2 g)(x), \quad x \in [0,\infty),$$

i.e. V_n commutes with the operator ψD^2 on $W_0^2(w(0,\gamma_\infty)\psi)$.

From Lemma 2.2 and Theorem 2.5 we get

Theorem 2.6. *If $f \in C(w(0,\gamma_\infty))$ and $m, n \in \mathbb{N}$, $m, n > -\gamma_\infty - 1$, then $V_n V_m f = V_m V_n f$, i.e. V_m and V_n commute on $C(w(0,\gamma_\infty))$.*

Proof. From Lemma 2.2 and Lemma 2.1 we observe that $V_j f \in W_0^2(w(0,\gamma_\infty)\psi)$, $j > -\gamma_\infty - 1$, whenever $f \in C(w(0,\gamma_\infty))$, i.e. we can apply Theorem 2.5 with $g = V_j f$. Set $\lambda_j = (j(j+1))^{-1}$. Without loss of generality we assume that $m = n + k$, $k \in \mathbb{N}$.

We prove the theorem by induction on k. For $k = 1$ using Lemma 2.2, (2.14) and Theorem 2.5 we get

$$
\begin{aligned}
&V_n^2 f - V_n V_{n+1} f = V_n(V_n f - V_{n+1} f) = V_n(\lambda_n \psi D^2(V_n f)) \\
&= \lambda_n V_n(\psi D^2(V_n f)) = \lambda_n \psi D^2(V_n^2 f) = (V_n - V_{n+1})V_n f = V_n^2 f - V_{n+1} V_n f,
\end{aligned}
$$

which gives $V_n V_{n+1} f = V_{n+1} V_n f$.

Assume $V_n V_{n+j} f = V_{n+j} V_n f$ for $j = 1, 2, \ldots, k$. Then using the inductive assumption, Lemma 2.2, (2.14) and Theorem 2.5 we get

$$\begin{aligned}
V_n^2 f - V_n V_{n+k+1} f &= V_n \sum_{j=0}^{k} (V_{n+j} f - V_{n+j+1} f) = V_n \sum_{j=0}^{k} \lambda_{n+j} \psi D^2 (V_{n+j} f) \\
&= \sum_{j=0}^{k} \lambda_{n+j} \psi D^2 (V_n V_{n+j} f) = \sum_{j=0}^{k} \lambda_{n+j} \psi D^2 (V_{n+j} V_n f) \\
&= \sum_{j=0}^{k} (V_{n+j} - V_{n+j+1}) V_n f = V_n^2 f - V_{n+k+1} V_n f
\end{aligned}$$

and, hence, $V_n V_{n+k+1} f = V_{n+k+1} V_n f$.This completes the proof. □

We finalize the section by proving the operators V_n and $\psi D^2 V_n$ have norm 1 in appropriate weighted norm spaces. The first lemma improves the constant c in Lemma 2.1 with a very simple proof but for different class of admissible weights.

Lemma 2.7. *Let w^{-1} be concave. Then for every $f \in C(w)$ and $n \in \mathbb{N}$ we have $\|wV_n f\| \le \|wf\|$, i.e. V_n has norm 1 as an operator from $C(w)$ to $C(w)$.*

Proof. From (2.14) and $w \ge 0$ we get

$$|V_n f(x)| = |V_n((wf)w^{-1})(x)| \le V_n(\|wf\| w^{-1})(x) = \|wf\| V_n(w^{-1})(x).$$

From the concavity of w^{-1} and (2.18) we get $V_n\left(w^{-1}\right) \le w^{-1}$, which proves the lemma. □

Lemma 2.8. *Let w^{-1} be concave. Then for every $g \in W^2(w\psi)$ and $n \in \mathbb{N}$ we have*

$$\|w\psi D^2(V_n g)\| \le \|w\psi D^2 g\|.$$

Proof. From the concavity of w^{-1} we get $w \ge cw(0,-1)$ for some positive constant c and hence $g \in W^2(w(0,-1)\psi)$. Applying the representation from Lemma 2.4 and the inequality $V_n\left(w^{-1}\right) \le w^{-1}$ for the concave function w^{-1} as in Lemma 2.7 we obtain

$$\begin{aligned}
&|w(x)\psi(x)D^2(V_n g)(x)| \\
&\qquad = w(x) \sum_{k=1}^{\infty} P_{n,k}(x) \int_0^\infty (n+1) P_{n+2,k-1}(y) \psi(y) D^2 g(y)\, dy \\
&\qquad \le \|w\psi D^2 g\|\; w(x) \sum_{k=1}^{\infty} P_{n,k}(x) \int_0^\infty (n+1) P_{n+2,k-1}(y) w^{-1}(y)\, dy \\
&\qquad \le \|w\psi D^2 g\|\; w(x) V_n\left(w^{-1})(x\right) \le \|w\psi D^2 g\|,
\end{aligned}$$

which proves the lemma. □

Note that Lemma 2.8 immediately follows from Theorem 2.5 and Lemma 2.7 if we assume $g \in W_0^2(w\psi)$.

3 Proof of the direct theorem

In the proof of the direct theorem we use the following two lemmas.

Lemma 3.1. *For every $g \in C(w(0,-2))$ and $x \in [0,\infty)$ we have*

$$\lim_{n\to\infty} V_n g(x) = g(x).$$

Proof. For $x = 0$ we have $V_n g(0) = g(0)$ for every $n \in \mathbb{N}$.

Now, fix $x > 0$ and $\varepsilon > 0$. We choose $\delta = \delta(x,\varepsilon)$, $\delta < 1$, so that $|g(x)-g(t)| \le \varepsilon/2$ for $|x-t| \le \delta$. Taking into account that $w(0,-2)(t)^{-1} \le (x+2)^2(x-t)^2\delta^{-2}$ for every $t \ge 0$, $|x-t| \ge \delta$ we get

$$|g(x) - g(t)| \le Q(t) := M(x-t)^2 + \varepsilon/2 \quad \forall\, t \in [0,\infty)$$

with $M = 2(x+2)^2\delta^{-2}\|wg\|$. Using the above inequality, (2.14), (2.15) and (2.16) we get

$$|g(x) - V_n g(x)| = |V_n(g(x) - g, x)| \le V_n(Q,x) \le M\frac{2}{n-1}\psi(x) + \varepsilon/2 \le \varepsilon$$

for every $n \ge 4M\psi(x)\varepsilon^{-1} + 1$. This proves the lemma. □

Lemma 3.1 represents a standard point-wise convergence statement for positive linear operators. As shown in [10, Theorem 3.1], the lemma is true for every g with no faster than power growth at infinity, i.e. $g \in C(w(0,\gamma_\infty))$ for some negative γ_∞, but the proof of this property requires the verification of the point-wise convergence of V_n on all polynomials (not only on the quadratic ones as provided by (2.15) and (2.16)).

The next lemma is a weighted Jackson-type inequality for the operators V_n.

Lemma 3.2. *Let w^{-1} be concave. If $g \in W^2(w\psi)$ then $g \in C(w(0,-1-\varepsilon))$ for every $\varepsilon > 0$ and*

$$\|w(V_n g - g)\| \le \frac{1}{n}\|w\psi D^2 g\|, \quad n \in \mathbb{N}.$$

Proof. From the concavity of w^{-1} we get $g \in W^2(w(0,-1)\psi)$. Now, in view of (2.23) we see that $g(x)$ grows at infinity no faster than $x \log x$. Hence $g \in C(w(0,-1-\varepsilon))$ for every positive ε and Lemma 3.1 gives $\lim_{n\to\infty} V_n g(x) = g(x)$ for every $x \in [0,\infty)$. Then from Lemma 2.2 with $\gamma_\infty = -1-\varepsilon$ we have

$$V_n g(x) - g(x) = \sum_{k=n}^{\infty}(V_k g(x) - V_{k+1}g(x)) = \sum_{k=n}^{\infty}\frac{\psi(x)D^2V_k g(x)}{k(k+1)}. \tag{3.1}$$

From (3.1) and Lemma 2.8 we get

$$\begin{aligned}\|w(V_n g - g)\| &= \left\| w\sum_{k=n}^{\infty}\frac{\psi D^2 V_k g}{k(k+1)}\right\| \\ &\le \sum_{k=n}^{\infty}\frac{\|w\psi D^2 V_k g\|}{k(k+1)} \le \sum_{k=n}^{\infty}\frac{1}{k(k+1)}\|w\psi D^2 g\| = \frac{1}{n}\|w\psi D^2 g\|.\end{aligned}$$

□

Proof of Theorem 1.2. Let g be an arbitrary function from $W^2(w\psi)$ such that $g - f \in C(w)$. Then

$$\|w(V_n f - f)\| \leq \|w(V_n f - V_n g)\| + \|w(V_n g - g)\| + \|w(g - f)\|.$$

From Lemma 2.7 and Lemma 3.2 we get

$$\|w(V_n f - f)\| \leq 2\|w(f - g)\| + \frac{1}{n}\|w\psi D^2 g\| = 2\left(\|w(f - g)\| + \frac{1}{2n}\|w\psi D^2 g\|\right).$$

Taking an infimum on $g \in W^2(w\psi)$ in the above inequality we prove the theorem. $\square$

4 Proof of the inverse theorem

We start with a lemma showing that for every $f \in C(w) + W^2(w\psi)$ the images $V_n f$, $n \in \mathbb{N}$, can be tested for (almost-)realization of the K-functional (1.4).

Lemma 4.1. *Let w^{-1} be concave. If $f \in C(w) + W^2(w\psi)$ then for every $n \in \mathbb{N}$ we have $f - V_n f \in C(w)$ and $V_n f \in W_0^2(w\psi)$.*

Proof. Let $f = f_1 + f_2$ with $f_1 \in C(w)$ and $f_2 \in W^2(w\psi)$. In view of Lemma 3.2 with $\epsilon = 1/2$ we have $f_2 \in C(w(0, 3/2))$ and hence $V_n f = V_n f_1 + V_n f_2$ for every $n \in \mathbb{N}$.

Lemma 2.7 implies $V_n f_1 \in C(w)$ and Lemma 3.2 gives $f_2 - V_n f_2 \in C(w)$. Hence $f - V_n f \in C(w)$.

From Lemma 2.2 and Lemma 2.7 we obtain $V_n f_1 \in W_0^2(w\psi)$, Lemma 2.4 shows that $V_n f_2$ satisfies (1.3) on the place of g and Lemma 2.8 implies $V_n f_2 \in W^2(w\psi)$. Therefore $V_n f \in W_0^2(w\psi)$ and the lemma is proved. $\square$

The following two lemmas are crucial in the proof of the inverse theorem. The first one is a strong Voronovskaya-type estimate.

Lemma 4.2. *Let w^{-1} be concave. Then for every $g \in W_0^2(w\psi)$ such that $\psi D^2 g \in W^2(w\psi)$ and for every $n \in \mathbb{N}$ we have*

$$\left\|w\left(V_n g - g - \frac{1}{n}\psi D^2\left(\frac{g + V_n g}{2}\right)\right)\right\| \leq \frac{1}{4n^2}\|w\psi D^2(\psi D^2 g)\|.$$

Proof. From (3.1) and Theorem 2.5 we derive the representation

$$\begin{aligned}
V_n g - g - \frac{1}{n}\psi D^2\left(\frac{g+V_n g}{2}\right)
&= \sum_{k=n}^{\infty}\frac{V_k(\psi D^2 g)}{k(k+1)} - \frac{1}{2n}\psi D^2 g - \frac{1}{2n}\psi D^2(V_n g)\\
&= \sum_{k=2n}^{\infty}\frac{V_k(\psi D^2 g)-\psi D^2 g}{k(k+1)} + \sum_{k=n}^{2n-1}\frac{V_k(\psi D^2 g)-V_n(\psi D^2 g)}{k(k+1)}\\
&= \sum_{k=2n}^{\infty}\frac{V_k(\psi D^2 g)-\psi D^2 g}{k(k+1)} + \sum_{k=n+1}^{2n-1}\sum_{s=n}^{k-1}\frac{V_{s+1}(\psi D^2 g)-V_s(\psi D^2 g)}{k(k+1)}
\end{aligned}$$

with the series convergent in $C(w)$. From this representation, Lemma 3.2, Lemma 2.2 and Lemma 2.8 we get

$$\begin{aligned}
&\left\| w\left(V_n g - g - \frac{1}{n}\psi D^2\left(\frac{g+V_n g}{2}\right)\right)\right\|\\
&\le \sum_{k=2n}^{\infty}\frac{\|w(V_k(\psi D^2 g)-\psi D^2 g)\|}{k(k+1)} + \sum_{k=n+1}^{2n-1}\sum_{s=n}^{k-1}\frac{\|w(V_{s+1}(\psi D^2 g)-V_s(\psi D^2 g))\|}{k(k+1)}\\
&\le \sum_{k=2n}^{\infty}\frac{\|w\psi D^2(\psi D^2 g)\|}{k^2(k+1)} + \sum_{k=n+1}^{2n-1}\sum_{s=n}^{k-1}\frac{\|w\psi D^2 V_s(\psi D^2 g)\|}{k(k+1)s(s+1)}\\
&\le A_n\|w\psi D^2(\psi D^2 g)\| \qquad (4.1)
\end{aligned}$$

with

$$A_n = \sum_{k=2n}^{\infty}\frac{1}{k^2(k+1)} + \sum_{k=n+1}^{2n-1}\frac{1}{k(k+1)}\sum_{s=n}^{k-1}\frac{1}{s(s+1)}.$$

Changing the order of summation in the double sum above and using

$$\begin{aligned}
\sum_{s=n}^{2n-2}\frac{1}{s(s+1)}\sum_{k=s+1}^{2n-1}\frac{1}{k(k+1)} &= \sum_{s=n}^{2n-2}\frac{1}{s(s+1)}\left(\frac{1}{s+1}-\frac{1}{2n}\right)\\
&= \sum_{s=n}^{2n-2}\frac{1}{s(s+1)^2} - \frac{n-1}{2n^2(2n-1)}
\end{aligned}$$

we get

$$\begin{aligned}
A_n &= \sum_{k=2n}^{\infty}\frac{1}{k^2(k+1)} + \sum_{s=n}^{2n-2}\frac{1}{s(s+1)^2} - \frac{n-1}{2n^2(2n-1)}\\
&< \sum_{k=2n}^{\infty}\frac{1}{(k-1)k(k+1)} + \frac{n+2}{n+1}\sum_{k=n+1}^{2n-1}\frac{1}{(k-1)k(k+1)} - \frac{n-1}{2n^2(2n-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=2n}^{\infty}\left(\frac{1}{k+1}-\frac{2}{k}+\frac{1}{k-1}\right) \\
&+ \frac{n+2}{2n+2}\sum_{k=n+1}^{2n-1}\left(\frac{1}{k+1}-\frac{2}{k}+\frac{1}{k-1}\right)-\frac{n-1}{2n^2(2n-1)} \\
&= \frac{2n^3+3n^2-3n+2}{4n^2(n+1)^2(2n-1)} < \frac{1}{4n^2},
\end{aligned}
$$

which in view of (4.1) proves the lemma. □

The next lemma is a weighted Bernstein-type inequality for the Baskakov-type operators. We estimate the action of ψD^2 on the *second* degree V_n^2 of the operator in order to get a smaller constant in the right-hand side. This constant is not exact.

Lemma 4.3. *Let $w = w(\gamma_0, \gamma_\infty)$ be given by (1.2) with $\gamma_0, \gamma_\infty \in [-1, 0]$. Then for every $F \in C_0(w)$ and for every $n \in \mathbb{N}$, $n \geq 4$, we have*

$$
\frac{1}{n}\|w\psi D^2(V_n^2 F)\| \leq \frac{5}{3}\|wF\|.
$$

Proof. For the second derivative of $g = V_n F$ we get from Lemma 2.2 and Lemma 2.7 that $D^2(V_n F) \in C(w\psi)$. Lemma 2.2 also implies that g satisfies (1.3) and, hence, $g \in W_0^2(w\psi) \subset W_0^2(w(0,-1)\psi)$. Applying Theorem 2.5 with this g, using (2.4) with $m = n, j = k-1$, (2.5) with $m = n, j = k$ and integration by parts we get for every $x \geq 0$

$$
\begin{aligned}
n^{-1}&\left|w(x)\psi(x)D^2 V_n(V_n F)(x)\right| = n^{-1}\left|w(x)V_n(\psi D^2 V_n F)(x)\right| \\
&= n^{-1}w(x)\left|\sum_{k=1}^{\infty}P_{n,k}(x)\int_0^\infty (n+1)P_{n+2,k-1}(y)\psi(y)D^2V_nF(y)dy\right| \\
&= w(x)\left|\sum_{k=1}^{\infty}P_{n,k}(x)\int_0^\infty \psi\left(\frac{k}{n}\right)P_{n,k}(y)\sum_{i=1}^{\infty}v_{n,i}(F)D^2P_{n,i}(y)dy\right| \\
&= w(x)\left|\sum_{k=1}^{\infty}P_{n,k}(x)\psi\left(\frac{k}{n}\right)\sum_{i=1}^{\infty}v_{n,i}(F)\int_0^\infty DP_{n,k}(y)DP_{n,i}(y)dy\right| \\
&= w(x)\left|\sum_{k=1}^{\infty}P_{n,k}(x)\psi\left(\frac{k}{n}\right)\sum_{i=1}^{\infty}v_{n,i}(F)\int_0^\infty P_{n,k}(y)\frac{k-ny}{\psi(y)}P_{n,i}(y)\frac{i-ny}{\psi(y)}dy\right| \\
&\leq S_n(\gamma_0,\gamma_\infty;x)\|wF\| \qquad (4.2)
\end{aligned}
$$

with

$$
\begin{aligned}
&S_n(\gamma_0,\gamma_\infty;x) \\
&= w(x)\sum_{k=1}^{\infty}P_{n,k}(x)\psi\left(\frac{k}{n}\right)\sum_{i=1}^{\infty}v_{n,i}\left(\frac{1}{w}\right)\int_0^\infty P_{n,k}(y)\frac{|k-ny|}{\psi(y)}P_{n,i}(y)\frac{|i-ny|}{\psi(y)}dy.
\end{aligned}
$$

The next three estimates follow from Hölder's inequality.

$$S_n(\gamma_0,\gamma_\infty;x) \le S_n(-1,\gamma_\infty;x)^{-\gamma_0} S_n(0,\gamma_\infty;x)^{1+\gamma_0}; \tag{4.3}$$
$$S_n(-1,\gamma_\infty;x) \le S_n(-1,-1;x)^{-\gamma_\infty} S_n(-1,0;x)^{1+\gamma_\infty}; \tag{4.4}$$
$$S_n(0,\gamma_\infty;x) \le S_n(0,-1;x)^{-\gamma_\infty} S_n(0,0;x)^{1+\gamma_\infty}. \tag{4.5}$$

Applying (4.4) and (4.5) in (4.3) we get

$$\begin{aligned} S_n(\gamma_0,\gamma_\infty;x) \le{}& S_n(-1,-1;x)^{\gamma_0\gamma_\infty} S_n(-1,0;x)^{-\gamma_0(1+\gamma_\infty)} \\ &\times S_n(0,-1;x)^{-\gamma_\infty(1+\gamma_0)} S_n(0,0;x)^{(1+\gamma_0)(1+\gamma_\infty)}. \end{aligned} \tag{4.6}$$

Inequalities (4.2) and (4.6) imply that it is enough to prove for every $x \ge 0$

$$S_n(\gamma_0,\gamma_\infty;x) \le \frac{5}{3} \tag{4.7}$$

in the four extreme cases $(\gamma_0,\gamma_\infty) = (0,0), (-1,0), (0,-1), (-1,-1)$ in order to establish the lemma. Applying Cauchy's inequality we get

$$S_n(\gamma_0,\gamma_\infty;x) \le w(x) \sum_{k=1}^{\infty} P_{n,k}(x)\psi\left(\frac{k}{n}\right) \sqrt{E_{n,k}(w)}\sqrt{F_{n,k}} \tag{4.8}$$

where the quantities $E_{n,k}(w)$, $F_{n,k}$ are given for $k \in \mathbb{N}$ by

$$\begin{aligned} E_{n,k}(w) &= \sum_{i=1}^{\infty} \int_0^{\infty} \psi(y)^{-2} P_{n,k}(y) P_{n,i}(y) v_{n,i}^2(w^{-1})\, dy, \\ F_{n,k} &= \sum_{i=1}^{\infty} \int_0^{\infty} \psi(y)^{-2} P_{n,k}(y)(k-ny)^2 P_{n,i}(y)(i-ny)^2\, dy. \end{aligned}$$

For the estimate of $F_{n,k}$ we use (2.10), (2.11), (2.5) and (2.13) and get

$$\begin{aligned} F_{n,k} &= \int_0^{\infty} \psi(y)^{-2} P_{n,k}(y)(k-ny)^2 \left(\sum_{i=1}^{\infty} P_{n,i}(y)(i-ny)^2\right) dy \\ &\le \int_0^{\infty} \psi(y)^{-2} P_{n,k}(y)(k-ny)^2 n\psi(y) dy \\ &= n\int_0^{\infty} (k-ny) d\, P_{n,k}(y) = n^2 \int_0^{\infty} P_{n,k}(y)\, dy = \frac{n^2}{n-1}. \end{aligned} \tag{4.9}$$

For the rest of the proof we establish (4.7) via (4.8), (4.9) and estimates of $E_{n,k}(w)$ separately in each of the four extreme cases of weight w.

(I) Let $\gamma_0 = \gamma_\infty = -1$. Here $w(x) = x^{-1}$ and $v_{n,i}(w^{-1}) = i/n$. From (2.11), (2.3), (2.4) and (2.13) we get

$$E_{n,k}(w) = \int_0^\infty \frac{P_{n,k}(y)}{\psi^2(y)} \sum_{i=1}^\infty P_{n,i}(y) \left(\frac{i}{n}\right)^2 dy$$
$$= \int_0^\infty \frac{P_{n,k}(y)}{\psi^2(y)} \left(y^2 + \frac{\psi(y)}{n}\right) dy = \int_0^\infty \frac{P_{n,k}(y)}{(1+y)^2}\, dy + \int_0^\infty \frac{P_{n,k}(y)}{n\psi(y)}\, dy$$
$$= \frac{n}{(n+k)(n+k+1)} + \frac{1}{k(n+k)}.$$

From the above result and (4.9) we obtain for $k \geq 1$

$$\frac{(n+k)^2}{n^2} E_{n,k}(w) F_{n,k}$$
$$\leq T_{n,k} := \frac{(n+k)^2}{n^2} \left(\frac{n}{(n+k)(n+k+1)} + \frac{1}{k(n+k)}\right) \frac{n^2}{n-1}$$
$$= \frac{n+1}{n-1} \frac{k+1}{k} \frac{n+k}{n+k+1} \leq \frac{2(n+1)^2}{(n-1)(n+2)}.$$

Applying the above result in (4.8) and using (2.10) we get for $n \geq 4$

$$S_n(-1,-1;x) \leq x^{-1} \sum_{k=1}^\infty P_{n,k}(x) \frac{k}{n} \sqrt{T_{n,k}} \leq \sqrt{\frac{2(n+1)^2}{(n-1)(n+2)}} \leq \sqrt{\frac{25}{9}} = \frac{5}{3},$$

which proves (4.7) in the case $\gamma_0 = \gamma_\infty = -1$.

(II) Let $\gamma_0 = 0, \gamma_\infty = -1$. Here $w(x) = (1+x)^{-1}$ and $v_{n,i}(w^{-1}) = (n+i)/n$. From (2.10) and (2.11) we get

$$\sum_{i=1}^\infty P_{n,i}(y) \left(\frac{n+i}{n}\right)^2 = (1+y)^2 + \frac{\psi(y)}{n} - \frac{1}{(1+y)^n} = \psi(y) \sum_{s=0}^{n+1} \frac{1}{(1+y)^s} + \frac{\psi(y)}{n}.$$

Hence

$$E_{n,k}(w) = \int_0^\infty \frac{P_{n,k}(y)}{\psi^2(y)} \sum_{i=1}^\infty P_{n,i}(y) \left(\frac{n+i}{n}\right)^2 dy$$
$$= \int_0^\infty \frac{P_{n,k}(y)}{\psi^2(y)} \left(\psi(y) \sum_{s=0}^{n+1} \frac{1}{(1+y)^s} + \frac{\psi(y)}{n}\right) dy$$
$$= \sum_{s=1}^{n+2} \int_0^\infty \frac{P_{n,k}(y)}{y(1+y)^s}\, dy + \int_0^\infty \frac{P_{n,k}(y)}{n\psi(y)}\, dy$$
$$= \frac{1}{k} \sum_{s=1}^{n+2} \frac{n(n+1)\dots(n+s-1)}{(n+k)(n+k+1)\dots(n+s-1+k)} + \frac{1}{k(n+k)}$$
$$\leq \frac{n}{k(n+k)} \sum_{s=1}^{n+2} \left(\frac{2n+1}{2n+1+k}\right)^{s-1} + \frac{1}{k(n+k)}$$

$$\leq \frac{n}{k(n+k)}\frac{2n+1+k}{k} + \frac{1}{k(n+k)} = \frac{1}{k^2}\left(n+1+\frac{n^2}{n+k}\right).$$

From the above estimate and (4.9) we obtain for $k \geq 1$

$$\frac{k^2}{n^2}E_{n,k}(w)F_{n,k}$$
$$\leq T_{n,k} := \frac{k^2}{n^2}\frac{1}{k^2}\left(n+1+\frac{n^2}{n+k}\right)\frac{n^2}{n-1} = \frac{1}{n-1}\left(n+1+\frac{n^2}{n+k}\right).$$

Taking into account that for every $n \geq 4$ the quantity $T_{n,k}$ is a decreasing function of k and that $T_{n,1}$ is a decreasing function of n we get $T_{n,k} \leq T_{4,1} = 41/15$. Applying the last inequality in (4.8) and using (2.10) we get for $n \geq 4$

$$S_n(0,-1;x) \leq (1+x)^{-1}\sum_{k=1}^{\infty}P_{n,k}(x)\frac{k+n}{n}\sqrt{T_{n,k}} < \sqrt{\frac{41}{15}},$$

which proves (4.7) in the case $\gamma_0 = 0$, $\gamma_\infty = -1$.

(III) Let $\gamma_0 = -1$, $\gamma_\infty = 0$. Here $w(x) = x^{-1}(x+1)$ and $v_{n,i}(w^{-1}) = i/(n+i+1)$. From (2.3), (2.10) and (2.11) we get

$$\sum_{i=1}^{\infty}P_{n,i}(y)\left(\frac{i}{n+i+1}\right)^2$$
$$= \frac{1}{(1+y)^2}\sum_{i=1}^{\infty}\frac{(n+i-1)(n+i-2)}{(n-1)(n-2)}\frac{i^2}{(n+i+1)^2}P_{n-2,i}(y)$$
$$\leq \frac{1}{(1+y)^2}\frac{n-2}{n-1}\sum_{i=1}^{\infty}P_{n-2,i}(y)\frac{i^2}{(n-2)^2} = \frac{1}{(1+y)^2}\frac{n-2}{n-1}\left(y^2+\frac{\psi(y)}{n-2}\right).$$

From this estimate, (2.3), (2.4) and (2.13) we get

$$\begin{aligned}
E_{n,k}(w) &= \int_0^\infty \frac{P_{n,k}(y)}{\psi^2(y)}\sum_{i=1}^{\infty}P_{n,i}(y)\left(\frac{i}{n+i+1}\right)^2 dy\\
&\leq \int_0^\infty \frac{P_{n,k}(y)}{\psi^2(y)}\frac{1}{(1+y)^2}\frac{n-2}{n-1}\left(y^2+\frac{\psi(y)}{n-2}\right)dy\\
&= \frac{n-2}{n-1}\int_0^\infty \frac{P_{n,k}(y)}{(1+y)^4}\,dy + \int_0^\infty \frac{P_{n,k}(y)}{(n-1)(1+y)^2\psi(y)}\,dy\\
&= \frac{(n-2)n(n+1)(n+2)}{(n-1)(n+k)(n+k+1)(n+k+2)(n+k+3)}\\
&\quad + \frac{n(n+1)(n+2)}{(n-1)k(n+k)(n+k+1)(n+k+2)}\\
&= \frac{n(n+1)(n+2)(nk-k+n+3)}{(n-1)k(n+k)(n+k+1)(n+k+2)(n+k+3)}.
\end{aligned} \tag{4.10}$$

From (2.2) and (2.3) we also get

$$\frac{(1+x)P_{n,k}(x)}{x}\psi\left(\frac{k}{n}\right)=P_{n,k-1}(x)\frac{(n+k)(n+k-1)}{n^2}. \tag{4.11}$$

Having in mind (4.11) we obtain from (4.10) and (4.9) for $k\geq 1$ and $n\geq 4$

$$\begin{aligned}
&\left(\frac{(n+k)(n+k-1)}{n^2}\right)^2 E_{n,k}(w)F_{n,k}\\
&\leq T_{n,k}:=\frac{(n+k)^2(n+k-1)^2\cdot n(n+1)(n+2)(nk-k+n+3)\cdot n^2}{n^4\cdot(n-1)k(n+k)(n+k+1)(n+k+2)(n+k+3)\cdot(n-1)}\\
&\qquad\qquad=\frac{(n+1)(n+2)(nk-k+n+3)(n+k)(n+k-1)^2}{n(n-1)(nk-k)(n+k+1)(n+k+2)(n+k+3)}\leq\frac{5}{2}.
\end{aligned}$$

Applying (4.11) and the above result in (4.8) and using (2.10) we get for $n\geq 4$

$$S_n(-1,0;x)\leq\sum_{k=1}^{\infty}P_{n,k-1}(x)\sqrt{T_{n,k}}\leq\sqrt{\frac{5}{2}},$$

which proves (4.7) in the case $\gamma_0=-1,\gamma_\infty=0$.

(IV) Let $\gamma_0=\gamma_\infty=0$. Here $w(x)=1$ and $v_{n,i}(w^{-1})=1$. From (2.10), the definition of Baskakov basic functions and (2.13) we get

$$\begin{aligned}
E_{n,k}(w)&=\int_0^\infty\frac{P_{n,k}(y)}{\psi^2(y)}\sum_{i=1}^{\infty}P_{n,i}(y)dy=\int_0^\infty\frac{P_{n,k}(y)}{\psi^2(y)}\left(1-(1+y)^{-n}\right)dy\\
&=\int_0^\infty\frac{P_{n,k}(y)}{\psi^2(y)}y\sum_{r=1}^{n}\frac{1}{(1+y)^r}dy\\
&=\sum_{r=1}^{n}\binom{n+k-1}{k}\binom{n+k+r+1}{k-1}^{-1}\int_0^\infty P_{n+r+3,k-1}(y)\,dy\\
&=\sum_{r=1}^{n}\binom{n+k-1}{k}\binom{n+k+r+1}{k-1}^{-1}\frac{1}{n+r+2}\\
&=\frac{n(n+1)(n+2)}{k(n+k)(n+k+1)(n+k+2)}Q_{n,k},
\end{aligned}\tag{4.12}$$

where

$$Q_{n,k}=1+\sum_{r=2}^{n}\left(\prod_{s=2}^{r}\frac{n+s+1}{n+k+s+1}\right).$$

A trivial estimate of the above quantity is $Q_{n,k}\leq n$. Another estimate is

$$Q_{n,k}\leq 1+\frac{n+3}{n+k+3}\sum_{r=2}^{n}\left(\frac{2n+1}{2n+1+k}\right)^{r-2}\leq 1+\frac{n+3}{n+k+3}\,\frac{2n+1+k}{k}. \tag{4.13}$$

From (4.13), (4.9) and (4.12) we get for $k \geq 1$ and $n \geq 4$

$$\begin{aligned}
&\psi\left(\frac{k}{n}\right)^2 E_{n,k}(w)F_{n,k} \\
&\qquad\leq T_{n,k} := \frac{k^2(n+k)^2 \cdot n(n+1)(n+2)Q_{n,k} \cdot n^2}{n^4 \cdot k(n+k)(n+k+1)(n+k+2) \cdot (n-1)} \\
&\qquad\leq \tilde{T}_{n,k} := \frac{(n+1)(n+2)(n+k)[(2n+k+1)(n+3)+k(n+k+3)]}{n(n-1)(n+k+1)(n+k+2)(n+k+3)}.
\end{aligned}$$

For every $n \geq 4$ the quantity $\tilde{T}_{n,k}$ is a decreasing function of k. Hence, $T_{n,k} \leq \tilde{T}_{n,k} \leq \tilde{T}_{n,1} \leq \tilde{T}_{5,1} = 21/8$ for $n \geq 5$ and $T_{4,k} \leq \tilde{T}_{4,k} \leq \tilde{T}_{4,3} = 133/48$. For $n = 4$ and $k = 1, 2$ we can improve the upper bound $\tilde{T}_{n,k}$ if we apply the trivial estimate $Q_{n,k} \leq n$ instead of (4.13). This leads to

$$T_{n,k} \leq \frac{(n+1)(n+2)(n+k)k}{(n-1)(n+k+1)(n+k+2)}$$

and, hence, $T_{4,1} \leq 25/21$ and $T_{4,2} \leq 15/7$. Thus, we obtain $T_{n,k} \leq 133/48$ for every $k \geq 1$ and $n \geq 4$.

Applying this estimate in (4.8) and using (2.10) we get for $n \geq 4$

$$S_n(0,0;x) \leq \sum_{k=1}^{\infty} P_{n,k}(x)\sqrt{T_{n,k}} < \sqrt{\frac{133}{48}},$$

which proves (4.7) in the case $\gamma_0 = \gamma_\infty = 0$ and completes the proof of the lemma. $\square$

Proof of Theorem 1.3. We follow the scheme for proving strong inverse theorems of type A given in [4]. From Lemma 4.1 we get that $f - V_n^k f \in C(w)$ and $V_n^k f \in W_0^2(w\psi)$ for every $k \in \mathbb{N}$. Applying Lemma 4.2 with $g = V_n^4 f$, Theorem 2.5 with $g = V_n^3 f$ and $g = V_n^2 f$ and Lemma 4.3 with $F = \psi D^2(V_n^2 f) \in C_0(w)$ we get

$$\begin{aligned}
\left\| w\left(V_n^5 f - V_n^4 f - \frac{1}{n}\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right)\right\| &\leq \frac{1}{4n^2}\left\|w\psi D^2\left(\psi D^2(V_n^4 f)\right)\right\| \\
&= \frac{1}{4n^2}\left\|w\psi D^2\left(V_n^2\left(\psi D^2(V_n^2 f)\right)\right)\right\| \leq \frac{5}{12n}\|w\psi D^2(V_n^2 f)\|.
\end{aligned}$$

Using the last inequality, Lemma 4.3 with $F = f - V_n^3 f \in C_0(w)$ and with

$F = f - V_n^2 f \in C_0(w)$ and Lemma 2.7 we get

$$\begin{aligned}
&\left\| w\left(V_n^5 f - V_n^4 f - \frac{1}{n}\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right)\right\| \\
&\leq \frac{5}{12n}\left\| w\psi D^2 V_n^2\left(f - \frac{V_n^2 f + V_n^3 f}{2}\right)\right\| + \frac{5}{12n}\left\| w\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right\| \\
&\leq \frac{5}{24n}\left\| w\psi D^2\left(V_n^2\left(f - V_n^2 f\right)\right)\right\| + \frac{5}{24n}\left\| w\psi D^2\left(V_n^2\left(f - V_n^3 f\right)\right)\right\| \\
&+ \frac{5}{12n}\left\| w\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right\| \\
&\leq \frac{25}{72}\left(\|w(V_n^2 f - f)\| + \|w(V_n^3 f - f)\|\right) + \frac{5}{12n}\left\| w\psi D^2\left(\frac{V_n^5 f + V_n^4 f}{2}\right)\right\| \\
&\leq \frac{125}{72}\|w(V_n f - f)\| + \frac{5}{12n}\left\| w\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right\|.
\end{aligned}$$

From the above inequality and Lemma 2.7 we have

$$\begin{aligned}
&\frac{1}{n}\left\| w\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right\| \\
&\leq \left\| w\left(V_n^5 f - V_n^4 f - \frac{1}{n}\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right)\right\| + \|w(V_n^5 f - V_n^4 f)\| \\
&\leq \frac{197}{72}\|w(V_n f - f)\| + \frac{5}{12n}\left\| w\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right\|,
\end{aligned}$$

which can be rewritten as

$$\frac{1}{2n}\left\| w\psi D^2\left(\frac{V_n^4 f + V_n^5 f}{2}\right)\right\| \leq \frac{197}{84}\|w(V_n f - f)\|. \tag{4.14}$$

Finally from the definition of K-functional, Lemma 2.7 and (4.14) we obtain

$$\begin{aligned}
K_w\left(f, \frac{1}{2n}\right) &= \inf\left\{\|w(f-g)\| + \frac{1}{2n}\|w\psi D^2 g\| \ : \ g \in W^2(w\psi)\right\} \\
&\leq \left\| w\left(f - \frac{V_n^4 f + V_n^5 f}{2}\right)\right\| + \frac{1}{2n}\left\| w\psi D^2\left(\frac{V_n^5 f + V_n^4 f}{2}\right)\right\| \\
&\leq \left(\frac{9}{2} + \frac{197}{84}\right)\|w(f - V_n f)\| = \frac{575}{84}\|w(f - V_n f)\|.
\end{aligned}$$

Theorem 1.3 is proved. □

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