Exact Constants in Estimations of the Error of the Quadrature Formulae of Simpson with the Averaged Moduli of Smoothness

I. A. Parvanova and P. E. Parvanov

February 17, 2007

Abstract

In this paper the exact constants in estimations of the error of the quadrature formulae of Simpson are obtained. The estimations involve 4−th order in the non-periodic case and high order in the periodic case L_1-averaged Sendov-Popov moduli of smoothness of a bounded and measurable function f.

1 Introduction.

Let $M[0,1]$ be the set of all bounded and measurable on $[0,1]$ functions and Π_k be the set of all algebraic polynomials of degree at most k. The $k-$th order local modulus of smoothness of $f \in M[0,1]$ at the point $x \in [0,1]$ with a step $\delta \in \left[0, \frac{1}{k}\right]$ is a function (see Definition 1.4 of [2])

$$\omega_k(f,x;\delta) \stackrel{\text{def}}{=} \sup\left\{|\Delta^k_{v,[0,1]} f(t)| \ ; \ t, t+kv \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right]\right\}, \tag{1.1}$$

where

$$\Delta^k_{t,[0,1]} f(x) \stackrel{\text{def}}{=} \begin{cases} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+it) & if \ \ x, x+kt \ \in [0,1]; \\ 0 & otherwise. \end{cases}$$

The k-th order L_1-averaged Sendov-Popov modulus of smoothness of a function f bounded and measurable on [0,1] is (see Definition 1.5 of [2])

$$\tau_k(f,\delta)_{L_1[0,1]} \stackrel{\text{def}}{=} \|\omega_k(f,\cdot;\delta)\|_{L_1[0,1]}. \tag{1.2}$$

We consider the well-known quadrature formulae of Simpson

$$\int_0^1 f(x)dx \approx Q_1^2(f) = \frac{1}{6}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right) \tag{1.3}$$

and the n-composite quadrature formulae of Simpson

$$\int_0^1 f(x)dx \approx Q_n^2(f) = \frac{h}{6}\left(f(0) + 2\sum_{i=1}^{n-1} f(x_i) + 4\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) + f(1)\right), \tag{1.4}$$

where $x_i = ih$ and $h = \frac{1}{n}$.

We estimate the error

$$R_n^2(f) \stackrel{\text{def}}{=} \left| Q_n^2(f) - \int_0^1 f(x)dx \right|. \tag{1.5}$$

Our aim is to find for some fixed k and $\alpha > 0$ constants $c_{k,\alpha}$ and $\tilde{c}_{k,\alpha}$ such that

$$c(k,\alpha) = \max_{f\in M[0,1],\, n\in\mathbb{N}} \left\{ \frac{R_n^2(f)}{\tau_k(f,\alpha h)_{L_1[0,1]}} \right\} \tag{1.6}$$

and

$$\tilde{c}(k,\alpha) = \sup_{f\in C^\infty[0,1],\, n\in\mathbb{N}} \left\{ \frac{R_n^2(f)}{\tau_k(f,\alpha h)_{L_1[0,1]}} \right\}. \tag{1.7}$$

In Section 2 we find the exact constants $c\left(4, \frac{1}{4}\right)$ and $\tilde{c}\left(4, \frac{1}{4}\right)$ for non-periodic functions. In Section 3 and 4 we consider periodic functions (1-periodic for simplicity). We find for this functions $c(k,\alpha)$ and $\tilde{c}(k,\alpha)$, where $k \in \mathbb{N}$ and $\alpha \geq \frac{1}{2}$ and $c\left(2, \frac{1}{4}\right)$ and $\tilde{c}\left(2, \frac{1}{4}\right)$. The methods in Section 3 and 4 are applicable for other $n-$composite quadrature formulae Q_n^s of Newton-Cotes type, constructed with respect to s knots.

2 Estimations of the error of the quadrature formulae for non-periodic functions.

In this section we consider non-periodic functions and a step $\alpha = \frac{1}{4}$. The Quadrature Formulae of Simpson is exact in Π_3. Then (see [1]) we can consider $k = 4$. We first prove the following lemma.

Lemma 2.1 *Let $f \in M[0,1]$ and $n \in \mathbb{N}$. Then*

$$R_n^2(f) \leq \frac{4}{9}\tau_4\left(f, \frac{1}{4n}\right)_{L_1[0,1]}.$$

Proof. Using (1.5) and (1.4) for $[a,b] = [0,1]$ we have

$$\begin{aligned} R_n^2(f) & \\ &= \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left(f(x) - \frac{1}{6n}\left(f(x_i) + 4\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_{i+1}) \right) \right) dx \right| \\ &= \left| \sum_{i=0}^{n} \frac{4}{6}\int_0^{\frac{h}{4}} \Delta_t^4 f(x_i)dt + \frac{4}{6}\int_0^{\frac{h}{4}} \Delta_{-t}^4 f(x_{i+1})dt + \frac{4}{9}\int_0^{\frac{h}{4}} \Delta_t^4 f\left(\frac{x_i+x_{i+1}}{2} - 2t\right)dt \right| \\ &\leq \sum_{i=0}^{n} \frac{4}{6}\left| \int_0^{\frac{h}{4}} \Delta_t^4 f(x_i)dt \right| + \frac{4}{6}\left| \int_0^{\frac{h}{4}} \Delta_{-t}^4 f(x_{i+1})dt \right| + \frac{4}{9}\left| \int_0^{\frac{h}{4}} \Delta_t^4 f\left(\frac{x_i+x_{i+1}}{2} - 2t\right)dt \right|. \end{aligned} \tag{2.1}$$

From (1.1) we have the next four inequalities are true for $t \in \left[0, \frac{h}{4}\right]$.

$$\left|\Delta_t^4 f(x_i)\right| \le \omega_4 \left(f, x_i + 2t; \frac{h}{4}\right); \tag{2.2}$$

$$\left|\Delta_{-t}^4 f(x_{i+1})\right| \le \omega_4 \left(f, x_{i+1} - 2t; \frac{h}{4}\right); \tag{2.3}$$

$$\left|\Delta_t^4 f\left(\frac{x_i + x_{i+1}}{2} - 2t\right)\right| \le \omega_4 \left(f, \frac{x_i + x_{i+1}}{2} + \frac{h}{2} - 2t; \frac{h}{4}\right); \tag{2.4}$$

$$\left|\Delta_t^4 f\left(\frac{x_i + x_{i+1}}{2} - 2t\right)\right| \le \omega_4 \left(f, \frac{x_i + x_{i+1}}{2} - \frac{h}{2} + 2t; \frac{h}{4}\right). \tag{2.5}$$

Applying the estimations of (2.2)-(2.5) in (2.1) we obtain

$$\begin{aligned}
R_n^2(f) &\le \frac{4}{6}\sum_{i=0}^{n}\int_0^{\frac{h}{4}} \omega_4\left(f, x_i + 2t; \frac{h}{4}\right) dt + \frac{4}{6}\sum_{i=0}^{n}\int_0^{\frac{h}{4}} \omega_4\left(f, x_{i+1} - 2t; \frac{h}{4}\right) dt \\
&+\frac{2}{9}\sum_{i=0}^{n}\int_0^{\frac{h}{4}} \omega_4\left(f, \frac{x_i + x_{i+1}}{2} - \frac{h}{2} + 2t; \frac{h}{4}\right) + \omega_4\left(f, \frac{x_i + x_{i+1}}{2} + \frac{h}{2} - 2t; \frac{h}{4}\right) dt \\
&= \sum_{i=0}^{n} \frac{1}{3}\int_{x_i}^{x_i+\frac{h}{2}} \omega_4\left(f, x; \frac{h}{4}\right) dx + \frac{1}{3}\int_{x_i+\frac{h}{2}}^{x_{i+1}} \omega_4\left(f, x; \frac{h}{4}\right) dx + \frac{1}{9}\int_{x_i}^{x_{i+1}} \omega_4\left(f, x; \frac{h}{4}\right) dx \\
&= \frac{4}{9}\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} \omega_4\left(f, x; \frac{h}{4}\right) dx \\
&= \frac{4}{9}\tau_4\left(f; \frac{h}{4}\right)_{L_1[0,1]}.
\end{aligned}$$

Lemma 2.1 is proved. ■

Let $n = 1$. We prove that the constant $\frac{4}{9}$ in Lemma 2.1 is exact. More then this constant is the solution of another problem. This is (see (1.6)) the constant $c\left(4, \frac{1}{4}\right)$ for non-periodic functions.

Lemma 2.2 *There is a function $f \in M[0,1]$ such that*

$$R_1^2(f) = \frac{4}{9}\tau_4\left(f, \frac{1}{4}\right)_{L_1[0,1]}.$$

Proof. Let $\mathbb{B}$ be the set of the binary rational numbers. For every natural number m we define

$$B_m \overset{\text{def}}{=} \left\{x \in \mathbb{B}\,;\; x = \frac{2s-1}{2^m},\; s = 0, ..., 2^{m-1}\right\}.$$

Let $c = \frac{15}{16}$. We consider the function

$$f(x) \overset{\text{def}}{=} \begin{cases} 1 & if\ \ x = 0\ or\ 1; \\ \frac{1}{6} & if\ \ x = \frac{1}{2}; \\ \frac{1}{48}c^{m-3} & if\ \ x \in B_m,\ m \ge 3; \\ 0 & if\ \ x = \frac{1}{4},\ \frac{3}{4}\ or\ x \notin \mathbb{B}. \end{cases}$$

Obviously $f \in M[0,1]$. We prove that

$$\sup\left\{\Delta_t^4 f(x)\ ;\ t \leq \frac{1}{4}\right\} \leq 1. \tag{2.6}$$

Let $x \in \left[0, \frac{1}{2}\right)$. There are 4 cases of finite differencies:
I)$\Delta^4_{\frac{x}{4}} f(0)$; II)$\Delta^4_{\frac{x}{2}} f(0)$; III)$\Delta^4_{\frac{1}{4}-\frac{x}{4}} f(x)$; IV)$\Delta^4_t f(x)$ for which 0, 1 and $\frac{1}{2}$ are not knots.

In IV) we have $\sup\left\{\Delta_t^4 f(x)\ ;\ t \leq \frac{1}{4}\right\} \leq 8\max\left\{f(x)\ ;\ x \neq 0,\ 1,\ \frac{1}{2}\right\} = \frac{8}{48} < 1$.

In I), II) and III) if $x \notin \mathbb{B}$ we have $\Delta_t^4 f(x) = 1$. Else (2.6) is equal to:
I)$1 - 8c^2 f(B_m) + 6cf(B_m) + f(B_m) \leq 1$;
II)$1 - 8cf(B_m) + 6f(B_m) + \frac{1}{c}f(B_m) \leq 1$;
III)$2f(B_m) - 8cf(B_m) + 1 \leq 1$.
The last three inequalities are true because of the choise of c. The situation for $x \in \left(\frac{1}{2}, 1\right]$ is the same. In this way we prove (2.6).

Hence $\omega_4\left(f, x; \frac{1}{4}\right) = \begin{cases} 1 & if\ x \neq \frac{1}{2}; \\ 3 & if\ x = \frac{1}{2}; \end{cases}$ and $\tau_4\left(f; \frac{1}{4}\right)_{L_1[0,1]} = 1$.

Applying (1.5) for this function we have $R_1^2(f) = \frac{4}{9}$. Then Lemma 2.2 is proved. ∎

Lemma 2.3 *There is a functional sequence* $\{f_m\}_{m=1}^{\infty}$, $f_m \in C^\infty[0,1]$ *such that*

$$\lim_{m\to\infty} \frac{R_1^2(f_m)}{\tau_4\left(f_m, \frac{1}{4}\right)_{L_1[0,1]}} = \frac{4}{9}.$$

Proof. Let f be the function of Lemma 2.2 and for $m = 1, ..., \infty$

$$g_m(x) \stackrel{\text{def}}{=} \begin{cases} e^{\frac{(mx)^2}{(mx)^2-1}} & if\ \ |mx| \leq 1; \\ \\ 0 & if\ \ |mx| > 1. \end{cases}$$

We set $f_m(x) \stackrel{\text{def}}{=} \sum_{s=0}^{2^m} f\left(\frac{s}{2^m}\right) g_{2^{2m+1}}\left(x - \frac{s}{2^m}\right)$.

For this functional sequence we have

$$f_m \in C^\infty[0,1],\ f_m(x) = f(x)\ \forall x \in \bigcup_{i=1}^{m} B_i\ and\ \lim_{m\to\infty} f_m(x) = f(x)\ \forall\, x \in [0,1].$$

Also for the error of the quadrature formulae and for the local modulus of f_m we have respectively $R_1^2(f_m) \geq \frac{4}{9} - \frac{1}{2^{2m-1}}$ and

$$\omega_4\left(f_m, x; \frac{1}{4}\right) = 1 + \left(6.\frac{15}{16} + 1\right)\frac{3^{m-4}5^{m-3}}{2^{4m-8}} = 1 + const.\left(\frac{15}{16}\right)^m,\ x \neq \frac{1}{2}.$$

Since

$$\tau_4\left(f_m; \frac{1}{4}\right)_{L_1[0,1]} = 1 + const.\left(\frac{15}{16}\right)^m.$$

Finally we obtain

$$\lim_{m\to\infty} \frac{R_1^2(f_m)}{\tau_4\left(f_m, \frac{1}{4}\right)_{L_1[0,1]}} = \frac{4}{9}$$

Lemma 2.3 is proved. ■

Summarizing the statements from Lemma 2.1, Lemma 2.2 and Lemma 2.3 we obtain

Theorem 2.1 *For non-periodic functions* $c\left(4, \frac{1}{4}\right) = \tilde{c}\left(4, \frac{1}{4}\right) = \frac{4}{9}$.

3 Estimations of the error of the quadrature formulae for 1-periodic functions and a step $\geq \frac{h}{2}$.

In this section we consider 1-periodic functions and a step $\alpha \geq \frac{1}{2}$. There are not periodic algebraic polynomials out of Π_0. Then (see [1]) we can make estimations with τ_k, for each $k \geq 1$. We first prove the following lemma.

Lemma 3.1 *Let* $f \in M[0,1]$ *be 1-periodic,* $k, n \in \mathbb{N}$ *and* $\alpha \geq \frac{1}{2}$. *Then*

$$R_n^2(f) \leq \frac{1}{\binom{k}{\left[\frac{k}{2}\right]}} \tau_k\left(f, \frac{\alpha}{n}\right)_{L_1[0,1]}.$$

Proof. We prove the statement for $\alpha = \frac{1}{2}$ because the modulus (1.2) is an increasing function of α.

1)Let $k = 2m$. Using (1.5) and (1.4) we have

$$R_n^2(f) = \left|\sum_{l=0}^{n-1} \frac{1}{3n} f(x_l) - \frac{1}{3}\int_{x_l-\frac{h}{2}}^{x_l+\frac{h}{2}} f(x)dx + \frac{2}{3n} f(x_l + \frac{h}{2}) - \frac{2}{3}\int_{x_l}^{x_{l+1}} f(x)dx\right|.$$

Applying in the last equality the trivial equality $\frac{2}{\binom{2m}{m}} \sum_{j=1}^{m} (-1)^j \binom{2m}{m-j} = -1$ we obtain

$$
\begin{aligned}
(3.1)\qquad & R_n^2(f) \\
&= \Bigg| \sum_{l=0}^{n-1} \frac{1}{3n} f(x_l) + \frac{2}{3\binom{2m}{m}} \sum_{j=1}^{m} (-1)^j \binom{2m}{m-j} \int_{x_l-\frac{h}{2}}^{x_l+\frac{h}{2}} f(x)dx \\
&+ \frac{2}{3n} f(x_l+\frac{h}{2}) + \frac{4}{3\binom{2m}{m}} \sum_{j=1}^{m} (-1)^j \binom{2m}{m-j} \int_{x_l}^{x_{l+1}} f(x)dx \Bigg| \\
&= \Bigg| \sum_{l=0}^{n-1} \frac{1}{3n} f(x_l) + \frac{2}{3\binom{2m}{m}} \sum_{j=1}^{m} \frac{(-1)^j \binom{2m}{m-j}}{j} j \sum_{l=0}^{n-1} \int_{x_l-\frac{h}{2}}^{x_l+\frac{h}{2}} f(x)dx \\
&+ \sum_{l=0}^{n-1} \frac{2}{3n} f(x_l+\frac{h}{2}) + \frac{4}{3\binom{2m}{m}} \sum_{j=1}^{m} \frac{(-1)^j \binom{2m}{m-j}}{j} j \sum_{l=0}^{n-1} \int_{x_l}^{x_{l+1}} f(x)dx \Bigg| \\
&= \Bigg| \sum_{l=0}^{n-1} \frac{1}{3n} f(x_l) + \frac{2}{3\binom{2m}{m}} \sum_{j=1}^{m} \frac{(-1)^j \binom{2m}{m-j}}{j} \sum_{i=0}^{n-1} \sum_{s=-j}^{j-1} \int_{x_i+s\frac{h}{2}}^{x_i+(s+1)\frac{h}{2}} f(x)dx \\
&+ \sum_{l=0}^{n-1} \frac{2}{3n} f(x_l+\frac{h}{2}) + \frac{4}{3\binom{2m}{m}} \sum_{j=1}^{m} \frac{(-1)^j \binom{2m}{m-j}}{j} \sum_{i=0}^{n-1} \sum_{s=-j}^{j-1} \int_{x_i+(s+1)\frac{h}{2}}^{x_i+(s+2)\frac{h}{2}} f(x)dx \Bigg| \\
&= \Bigg| \sum_{i=0}^{n-1} \frac{1}{3n} f(x_i) + \sum_{i=0}^{n-1} \frac{2}{3\binom{2m}{m}} \sum_{j=1}^{m} \frac{(-1)^j \binom{2m}{m-j}}{j} \int_{x_i-j\frac{h}{2}}^{x_i+j\frac{h}{2}} f(x)dx \\
&+ \sum_{i=0}^{n-1} \frac{2}{3n} f(x_i+\frac{h}{2}) + \sum_{i=0}^{n-1} \frac{4}{3\binom{2m}{m}} \sum_{j=1}^{m} \frac{(-1)^j \binom{2m}{m-j}}{j} \int_{x_i+\frac{h}{2}-j\frac{h}{2}}^{x_i+\frac{h}{2}+j\frac{h}{2}} f(x)dx \Bigg| \\
&= \Bigg| \sum_{i=0}^{n-1} \frac{2}{3\binom{2m}{m}} \int_0^{\frac{h}{2}} \Delta_t^{2m} f(x_i-mt)dt + \sum_{i=0}^{n-1} \frac{4}{3\binom{2m}{m}} \int_0^{\frac{h}{2}} \Delta_t^{2m} f(x_i+\frac{h}{2}-mt)dt \Bigg| \\
&\le \sum_{i=0}^{n-1} \frac{2}{3\binom{2m}{m}} \int_0^{\frac{h}{2}} |\Delta_t^{2m} f(x_i-mt)|dt + \sum_{i=0}^{n-1} \frac{4}{3\binom{2m}{m}} \int_0^{\frac{h}{2}} |\Delta_t^{2m} f(x_i+\frac{h}{2}-mt)|dt.
\end{aligned}
$$

From (1.1) we have the next four inequalities are true for $t \in \left[0, \frac{h}{2}\right]$.

$$\left|\Delta_t^{2m} f(x_i - mt)\right| \le \omega_{2m}\left(f, x_i + mt - m\frac{h}{2}; \frac{h}{2}\right); \qquad (3.2)$$

$$\left|\Delta_t^{2m} f(x_i - mt)\right| \le \omega_{2m}\left(f, x_i - mt + m\frac{h}{2}; \frac{h}{2}\right); \qquad (3.3)$$

$$\left|\Delta_t^{2m} f\left(x_i + \frac{h}{2} - mt\right)\right| \le \omega_{2m}\left(f, x_i + \frac{h}{2} + mt - m\frac{h}{2}; \frac{h}{2}\right); \qquad (3.4)$$

$$\left|\Delta_t^{2m} f\left(x_i + \frac{h}{2} - mt\right)\right| \le \omega_{2m}\left(f, x_i + \frac{h}{2} - mt + m\frac{h}{2}; \frac{h}{2}\right). \qquad (3.5)$$

Applying the estimations of (3.2)-(3.5) in (3.1) we obtain

$$
\begin{aligned}
(3.6)\qquad R_n^2(f) &\le \sum_{l=0}^{n-1} \frac{1}{3\binom{2m}{m}} \int_0^{\frac{h}{2}} \left(\omega_{2m}\left(f, x_i + mt - m\frac{h}{2}; \frac{h}{2}\right)\right.\\
&\left.+\omega_{2m}\left(f, x_i - mt + m\frac{h}{2}; \frac{h}{2}\right)\right) dt\\
&+\sum_{i=0}^{n-1} \frac{2}{3\binom{2m}{m}} \int_0^{\frac{h}{2}} \left(\omega_{2m}\left(f, x_i + \frac{h}{2} + mt - m\frac{h}{2}; \frac{h}{2}\right)\right.\\
&\left.+\omega_{2m}\left(f, x_i + \frac{h}{2} - mt + m\frac{h}{2}; \frac{h}{2}\right)\right) dt\\
&= \sum_{l=0}^{n-1} \frac{1}{3m\binom{2m}{m}} \left(\int_{x_i - m\frac{h}{2}}^{x_i + m\frac{h}{2}} \omega_{2m}\left(f, x; \frac{h}{2}\right) dx + 2\int_{x_i + \frac{h}{2} - m\frac{h}{2}}^{x_i + \frac{h}{2} + m\frac{h}{2}} \omega_{2m}\left(f, x; \frac{h}{2}\right) dx\right)\\
&= \sum_{l=0}^{n-1} \frac{1}{\binom{2m}{m}} \int_{x_i}^{x_{i+1}} \omega_{2m}\left(f, x; \frac{h}{2}\right) dx\\
&= \frac{1}{\binom{2m}{m}} \tau_{2m}\left(f; \frac{h}{2}\right)_{L_1[0,1]}.
\end{aligned}
$$

2)Let $k = 2m+1$. Using the same arguments as in 1) we have

$$
\begin{aligned}
(3.7)\quad & R_n^2(f)\\
&= \left|\sum_{i=0}^{n-1} \frac{2}{3\binom{2m+1}{m}} \left(-\int_0^{\frac{h}{2}} \Delta_t^{2m+1} f(x_i - mt)dt + \int_0^{\frac{h}{2}} \Delta_t^{2m+1} f(x_i - (m+1)t)dt\right)\right.\\
&\left.+\sum_{i=0}^{n-1} \frac{4}{3\binom{2m+1}{m}} \left(-\int_0^{\frac{h}{2}} \Delta_t^{2m+} f(x_i + \frac{h}{2} - mt)dt + \int_0^{\frac{h}{2}} \Delta_t^{2m+} f(x_i + \frac{h}{2} - (m+1)t)dt\right)\right|\\
&= \sum_{i=0}^{n-1} \frac{2}{3\binom{2m+1}{m}} \left(\int_0^{\frac{h}{2}} \left|\Delta_t^{2m+1} f(x_i - mt)\right| dt + \int_0^{\frac{h}{2}} \left|\Delta_t^{2m+1} f(x_i - (m+1)t)\right| dt\right)\\
&+\sum_{i=0}^{n-1} \frac{4}{3\binom{2m+1}{m}} \left(\int_0^{\frac{h}{2}} \left|\Delta_t^{2m+} f(x_i + \frac{h}{2} - mt)\right| dt + \int_0^{\frac{h}{2}} \left|\Delta_t^{2m+} f(x_i + \frac{h}{2} - (m+1)t)\right| dt\right).
\end{aligned}
$$

Also as in 1) from (1.1) we have the next four inequalities are true for $t \in \left[0, \frac{h}{2}\right]$.

$$(3.8)\qquad \left|\Delta_t^{2m+1} f(x_i - mt)\right| \le \omega_{2m+1}\left(f, x_i + mt - m\frac{h}{2}; \frac{h}{2}\right);$$

$$(3.9)\qquad \left|\Delta_t^{2m+1} f(x_i - (m+1)t)\right| \le \omega_{2m+1}\left(f, x_i - mt + m\frac{h}{2}; \frac{h}{2}\right);$$

$$(3.10)\qquad \left|\Delta_t^{2m+1} f\left(x_i + \frac{h}{2} - mt\right)\right| \le \omega_{2m+1}\left(f, x_i + \frac{h}{2} + mt - m\frac{h}{2}; \frac{h}{2}\right);$$

$$(3.11)\qquad \left|\Delta_t^{2m+1} f\left(x_i+\frac{h}{2}-(m+1)t\right)\right| \le \omega_{2m+1}\left(f, x_i+\frac{h}{2}-mt+m\frac{h}{2};\frac{h}{2}\right).$$

Applying the estimatoins of (3.8)-(3.11) in (3.7) we obtain

$$(3.12)\qquad R_n^2(f) \le \sum_{l=0}^{n-1}\frac{1}{3\binom{2m+1}{m}}\int_0^{\frac{h}{2}}\omega_{2m+1}\left(f,x_i+mt;\frac{h}{2}\right)+\omega_{2m+1}\left(f,x_i-mt;\frac{h}{2}\right)dt$$
$$+\sum_{i=0}^{n-1}\frac{2}{3\binom{2m+1}{m}}\int_0^{\frac{h}{2}}\omega_{2m+1}\left(f,x_i+\frac{h}{2}+mt;\frac{h}{2}\right)+\omega_{2m+1}\left(f,x_i+\frac{h}{2}-mt;\frac{h}{2}\right)dt$$
$$=\sum_{l=0}^{n-1}\frac{1}{3m\binom{2m+1}{m}}\left(\int_{x_i-m\frac{h}{2}}^{x_i+m\frac{h}{2}}\omega_{2m+1}\left(f,x;\frac{h}{2}\right)dx+2\int_{x_i+\frac{h}{2}-m\frac{h}{2}}^{x_i+\frac{h}{2}+m\frac{h}{2}}\omega_{2m+1}\left(f,x;\frac{h}{2}\right)\right)dx$$
$$=\sum_{l=0}^{n-1}\frac{1}{\binom{2m+1}{m}}\int_{x_i}^{x_{i+1}}\omega_{2m+1}\left(f,x;\frac{h}{2}\right)dx$$
$$=\frac{1}{\binom{2m+1}{m}}\tau_{2m+1}\left(f;\frac{h}{2}\right)_{L_1[0,1]}.$$

The inequalities (3.6) and (3.12) complete the proof of Lemma 3.1. ■

Lemma 3.2 *Let $k,n\in\mathbb{N}$ and $\alpha\ge\frac{1}{2}$. There is a 1-periodic function $f\in M[0,1]$ such that*

$$R_n^2(f)=\frac{1}{\binom{k}{[\frac{k}{2}]}}\tau_k\left(f,\frac{\alpha}{n}\right)_{L_1[0,1]}.$$

Proof. There are many trivial bounded and measurable 1-periodic functions which prove this lemma but we construct a such function which we use in the next lemma. We use notations $\mathbb{B}$ and B_m from Lemma 2.1. We set $q_k \overset{\text{def}}{=} \dfrac{2^{k-1}-\binom{k}{[\frac{k}{2}]}}{\binom{k}{[\frac{k}{2}]}}<1$, $z_k\overset{\text{def}}{=}\left[\dfrac{k}{4}\right]+1$ and $\diamondsuit_{k,l}\overset{\text{def}}{=}\cup_{i=(l-1)z_k+1}^{lz_k}B_i,\ l\ge 1$. Let $n=1$ (for $n>1$ the idea is the same if we assume that $[x_i,x_{i+1}]=[0,1]$). We define the function

$$f(x)\overset{\text{def}}{=}\begin{cases}(q_k)^{l-1} & if\ \ x\in\diamondsuit_{k,l};\\ 0 & if\ \ x\notin\mathbb{B}.\end{cases}$$

Obviously $f\in M[0,1]$. For the finite difference $\Delta_v^k f(x)$ with $v\le\frac{1}{2n}$ and $x\in[0,1]$ we have 3 cases:
I)The knots of the finite difference are binary rational;
II)The knots of the finite difference are not binary rational;
III)The knots of the finite difference accept one are not binary rational.
In I) the absolute value of the finite difference is $<\binom{k}{[\frac{k}{2}]}$. In II) we have that the value of this

difference is 0. In III) the absolute value of the finite difference is $\leq \binom{k}{[\frac{k}{2}]}$ ("=" iff $f(x) = 1$). Sience $\omega_k(f, x; \frac{\alpha}{n}) = \binom{k}{[\frac{k}{2}]}$ and $\tau_k\left(f, \frac{\alpha}{n}\right)_{L_1[0,1]} = \binom{k}{\left[\frac{k}{2}\right]}$. From other hand applying (1.5) for this function we have $R_n^2(f) = 1$.

Then Lemma 3.2 is proved. ■

Lemma 3.3 *Let $k, n \in \mathbb{N}$ and $\alpha \geq \frac{1}{2}$. There is a sequence of 1-periodic functions $\{f_m\}_{m=1}^{\infty}$, $f_m \in C^{\infty}[0,1]$ such that*

$$\lim_{m\to\infty} \frac{R_n^2(f_m)}{\tau_k\left(f_m, \frac{\alpha}{n}\right)_{L_1[0,1]}} = \frac{1}{\binom{k}{[\frac{k}{2}]}}.$$

Proof. Let f be the function of Lemma 3.2 and for $m = 1, ..., \infty$ $g_m(x)$ be the functions of Lemma 2.2. We set $f_m(x) \stackrel{\text{def}}{=} \sum_{s=0}^{2^{mz_k}} f\left(\frac{s}{2^{mz_k}}\right) g_{2^{2mz_k+1}}\left(x - \frac{s}{2^{mz_k}}\right)$.

For this functional sequence we have $f_m \in C^{\infty}[0,1]$, $f_m(x) = f(x)$ for every $x \in \cup_{i=1}^{mz_k} B_i$ and

$\lim_{m\to\infty} f_m(x) = f(x)$ for every $x \in [0,1]$.

Also for the error of the quadrature formulae and for the k-th averaged modulus of f_m we have respectively $\lim_{m\to\infty} R_n^2(f_m) = 1$ and $\lim_{m\to\infty} \tau_k\left(f_m, \frac{\alpha}{n}\right)_{L_1[0,1]} = \binom{k}{\left[\frac{k}{2}\right]}$.

Lemma 3.3 is proved. ■

Summarizing the statements from Lemma 3.1, Lemma 3.2 and Lemma 3.3 we obtain

Theorem 3.1 *Let $k \in \mathbb{N}$ and $\alpha \geq \frac{1}{2}$. For 1-periodic functions*

$$c(k, \alpha) = \tilde{c}(k, \alpha) = \frac{1}{\binom{k}{[\frac{k}{2}]}}.$$

Remark. Obviously the methods in Section 3 and 4 are applicable for other $n-$composite quadrature formulae Q_n^s of Newton-Cotes type, constructed with respect to s knots. Let k, n, $s \in \mathbb{N}$ and $\alpha \geq \frac{1}{s}$. For 1-periodic functions we can prove the following estimation

$$R_n^s(f) \leq \frac{1}{\binom{k}{[\frac{k}{2}]}} \tau_k\left(f, \frac{\alpha}{n}\right)_{L_1[0,1]},$$

where the constant is exact in $M[0,1]$ and $C^{\infty}[0,1]$.

4 Estimations of the error of the quadrature formulae for 1-periodic functions and a step $< \frac{h}{2}$.

In this section we consider 1-periodic functions and a step $\alpha < \frac{1}{2}$. There are some specific problems. Here we use not only finite differencies centered in the knots of the quadrature formulae. We demonstrate this for $k = 2$ and $\alpha = \frac{1}{4}$. We first prove the upper estimation.

Lemma 4.1 *Let $f \in M[0,1]$ be 1-periodic and $n \in \mathbb{N}$. Then*

$$R_n^2(f) \le \frac{2}{3}\tau_2\left(f, \frac{1}{4n}\right)_{L_1[0,1]}.$$

Proof. Using again (1.5) and (1.4) we have

$$\begin{aligned}
(4.1)\qquad R_n^2(f) &= \left|\sum_{i=0}^{n-1}\frac{1}{3}\int_0^{\frac{h}{4}}\Delta_t^2 f(x_i-t)dt + \sum_{i=0}^{n-1}\frac{2}{3}\int_0^{\frac{h}{4}}\Delta_t^2 f\left(x_i+\frac{h}{2}-t\right)dt\right.\\
&\left.-\sum_{i=0}^{n-1}\frac{1}{3}\int_0^{\frac{h}{2}}\Delta_{\frac{h}{4}}^2 f\left(x_i-\frac{h}{4}+t\right)dt\right|\\
&\le \sum_{i=0}^{n-1}\frac{1}{3}\int_0^{\frac{h}{4}}\left|\Delta_t^2 f(x_i-t)\right|dt + \sum_{i=0}^{n-1}\frac{2}{3}\int_0^{\frac{h}{4}}\left|\Delta_t^2 f\left(x_i+\frac{h}{2}-t\right)\right|dt\\
&+\sum_{i=0}^{n-1}\frac{1}{3}\int_0^{\frac{h}{2}}\left|\Delta_{\frac{h}{4}}^2 f\left(x_i-\frac{h}{4}+t\right)\right|dt.
\end{aligned}$$

Again as in the previous sections from (1.1) we have that the next tree inequalities are true.

$$(4.2)\qquad \int_0^{\frac{h}{4}}\left|\Delta_t^2 f(x_i-t)\right|dt \le \frac{1}{2}\int_{x_i-\frac{h}{4}}^{x_i+\frac{h}{4}}\omega_2\left(f,x;\frac{h}{4}\right)dx;$$

$$(4.3)\qquad \int_0^{\frac{h}{4}}\left|\Delta_t^2 f\left(x_i+\frac{h}{2}-t\right)\right|dt \le \frac{1}{2}\int_{x_i+\frac{h}{4}}^{x_i+3\frac{h}{4}}\omega_2\left(f,x;\frac{h}{4}\right)dx;$$

$$(4.4)\qquad \int_0^{\frac{h}{2}}\left|\Delta_{\frac{h}{4}}^2 f\left(x_i-\frac{h}{4}+t\right)\right|dt \le \int_{x_i-\frac{h}{4}}^{x_i+\frac{h}{4}}\omega_2\left(f,x;\frac{h}{4}\right)dx.$$

Applying the estimatoins of (4.2)-(4.4) in (4.1) we obtain

$$R_n^2(f) \le \sum_{i=0}^{n-1}\frac{2}{3}\int_{x_i}^{x_{i+1}}\omega_2\left(f,x;\frac{h}{4}\right)dx = \frac{2}{3}\tau_2\left(f,\frac{1}{4n}\right)_{L_1[0,1]}.$$

Lemma 4.1 is proved. ■

Lemma 4.2 *Let $n \in \mathbb{N}$. There is a 1-periodic function $f \in M[0,1]$ such that*

$$R_n^2(f) = \frac{2}{3}\tau_2\left(f, \frac{1}{4n}\right)_{L_1[0,1]}.$$

Let $n = 1$ (for $n > 1$ the idea is the same if we assume that $[x_i, x_{i+1}] = [0,1]$). We define the function

$$f(x) \overset{\text{def}}{=} \begin{cases} 1 & if\ \ x = 0\ or\ 1;\\ 3 & if\ \ x = \frac{1}{2};\\ 2 & if\ \ x \in \left[\frac{1}{4},\frac{1}{2}\right)\cup\left(\frac{1}{2},\frac{3}{4}\right];\\ 0 & if\ \ x \in \left(0,\frac{1}{4}\right)\cup\left(\frac{4}{4},1\right). \end{cases}$$

Obviously $f \in M[0,1]$, $R_n^2(f) = \frac{4}{3}$ and $\tau_2\left(f, \frac{1}{4n}\right)_{L_1[0,1]} = 2$.

Lemma 4.2 is proved. ■

Lemma 4.3 *There is a sequence of 1-periodic functions* $\{f_m\}_{m=1}^{\infty}$, $f_m \in C^\infty[0,1]$ *such that*

$$\lim_{m\to\infty} \frac{R_n^2(f_m)}{\tau_2\left(f_m, \frac{1}{4n}\right)_{L_1[0,1]}} = \frac{2}{3}.$$

Proof. Let f be the function of Lemma 4.2 and for $m = 1, ..., \infty$ $g_m(x)$ be the functions of Lemma 2.2. We set

$$f_m(x) \stackrel{\text{def}}{=} \begin{cases} 2 + g_m\left(x - \frac{1}{2}\right) & if \;\; x \in \left[\frac{1}{4}, \frac{3}{4}\right]; \\ 2g_m\left(x - \frac{1}{4}\right) + g_m(x) & if \;\; x \in \left[0, \frac{1}{4}\right); \\ 2g_m\left(x - \frac{3}{4}\right) + g_m(1-x) & if \;\; x \in \left(\frac{3}{4}, 1\right]. \end{cases}$$

For this functional sequence we have $f_m \in C^\infty[0,1]$, $f_m(x) = f(x)$ if $x = 0, \dfrac{1}{4}, \dfrac{1}{2}, \dfrac{3}{4}$ *or* 1 and $\lim\limits_{m\to\infty} f_m(x) = f(x)$ for every $x \in [0,1]$.

Also for the error of the quadrature formulae and for the 2-th averaged modulus of f_m we have respectively $\lim\limits_{m\to\infty} R_n^2(f_m) = \dfrac{4}{3}$ and $\lim\limits_{m\to\infty} \tau_2\left(f_m, \dfrac{1}{4n}\right)_{L_1[0,1]} = 2$.

Lemma 4.3 is proved. ■

Summarizing again statements from Lemma 4.1, Lemma 4.2 and Lemma 4.3 we obtain

Theorem 4.1 *For 1-periodic functions* $c\left(2, \frac{1}{4}\right) = \tilde{c}\left(2, \frac{1}{4}\right) = \frac{2}{3}$.

References

[1] K. Ivanov. New estimates of errors of quadrature formulae, formulae of numerical differentiation and interpolation. *J. Analysis Mathematica, Hung. Acad. of Sci. Budapest,* **vol. 6**, (1980) 281-303.

[2] Bl. Sendov and V.A.Popov. The Averaged Moduli of Smoothness. *John Willey & Sons.* (1988).

Irena Parvanova and Parvan Parvanov
Department of Mathematics
Higher Transport School
1754 Slatina, Sofia, Bulgaria