Weighted Approximation By A Class Of Bernstein-type Operators

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Abstract

Direct theorem in terms of the weighted K-functional for the uniform weighted approximation errors of a class of Bernstein-type operators are obtained for functions from $C(w)[0,1]$ with weight of the form $x^{\gamma_0}(1-x)^{\gamma_1}$ for $\gamma_0, \gamma_1 \in [-1, 0]$.

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1 Introduction

The class of Bernstein-type operators discussed in this paper are given for natural n by

$$\tilde{B}_n(f, x) = \sum_{k=0}^{n} b_{n,k}(f) P_{n,k}(x),$$

where $P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and the functionals $b_{n,k}(f)$ satisfy the following conditions

(1.1) $\quad b_{n,0}(f) = f(0)$ and $b_{n,n}(f) = f(1)$;

(1.2) $\quad b_{n,k}(f)$ are linear and positive;

(1.3) $\quad \tilde{B}_n(e_i, x) = e_i(x)$ for i=0 and i=1;

(1.4) $\quad \tilde{B}_n(e_2, x) = e_2(x) + \alpha(n) x(1-x)$.

Here e_i (for $i = 0, 1, 2$) are the functions $e_i(x) = x^i$.

The functional $b_{n,k}(f)$ for $1 \le k \le n-1$ in the operators $\tilde{B}_n$ takes place of $f\left(\frac{k}{n}\right)$ in the classical Bernstein operators [4].

Denote the weight function by

$$w(x) = w(\gamma_0, \gamma_1; x) = x^{\gamma_0}(1-x)^{\gamma_1} \text{ for } x \in (0,1) \text{ and real } \gamma_0, \gamma_1. \tag{1.5}$$

Our main results will concern the values of the powers γ_0, γ_1 in the range $[-1, 0]$. By $\varphi(x) = x(1-x)$ we denote the other weight which is naturally connected with the second derivatives of operators and the error for the function $e_2(x)$. By $D = \frac{d}{dx}$ we denote the first derivative operator.

Let $C(0,1)$ be the space of all continuous functions bounded on (0,1) and let $C(w)(0,1) = \{f : wf \in C(0,1)\}$. The norm in $C(w)(0,1)$ is given by $\|f\|_{C(w)(0,1)} = \sup_{x\in(0,1)} |w(x)f(x)|$. The cases of (weighted) continuity at the end-points of the domain are denoted by $[0,1]$ on the place of $(0,1)$, namely

$$C(w)[0,1] = \left\{ f \in C(w)(0,1) : \exists \lim_{x\to 0+0} w(x)f(x) \text{ and } \lim_{x\to 1-0} w(x)f(x) \right\},$$
$$C_0(w)[0,1] = \left\{ f \in C(w)[0,1] : \lim_{x\to 0+0} w(x)f(x) = \lim_{x\to 1-0} w(x)f(x) = 0 \right\}.$$

The space of smooth functions considered in the paper is given by

$$W^2(w\varphi)(0,1) = \left\{ g, g' \in AC_{loc}(0,1) : w\varphi D^2 g \in L_\infty(0,1) \right\},$$

where $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,1)$ and $L_\infty(0,1)$ denotes the Lebesgue measurable and essentially bounded in $(0,1)$ functions.

In this paper we estimate the rate of weighted approximation by $\tilde{B}_n$ for functions in $C_0(w)[0,1] + \pi_1$, where π_1 is the set of all algebraical polynomials of degree 1. This space serves as a natural generalization on $C[0,1]$ for the unweighted case because $C[0,1] = C_0[0,1] + \pi_1$.

The weighted approximation error will be compared with the K-functional which for every $f \in C(w)(0,1)$ and $t > 0$ is defined by

$$K_w(f,t) = \inf \left\{ \|w(f-g)\| + t\|w\varphi D^2 g\| : g \in W^2(w\varphi)(0,1) \right\}. \tag{1.6}$$

Our main result is a direct inequality. It is a generalization of the result in [3], which treats the case $w = 1$ and Goodman-Sharma operator ([1] and [2]).

Theorem 1.1. *Let w be given by* (1.5) *with $\gamma_0, \gamma_1 \in [-1, 0]$. Then for every $f \in C_0(w)[0,1] + \pi_1$ and every $n \in \mathbb{N}$ we have*

$$\|w(\tilde{B}_n f - f)\| \le 2K_w\left(f, \frac{\alpha(n)}{2}\right)$$

Some remarks:

(1.) Both sides of Theorem 1.1 do not change if f is replaced by $f-q$ for any $q\in\pi_1$. Hence, it is enough to prove Theorem 1.1 for functions $f\in C_0(w)[0,1]$.

(2.) Functions from $C(w)[0,1]\backslash(C_0(w)[0,1]+\pi_1)$ are not considered in Theorem 1.1 because neither $\|w(f-U_nf)\|\to 0$ nor $K_w\left(f,n^{-1}\right)\to 0$ when $n\to\infty$ for such functions.

(3.) We consider $\gamma_0,\gamma_1\geq -1$ because functions $\tilde{B}_n(f)\in C_0(w)[0,1]$ with $\gamma_0,\gamma_1=-1$.

(4.) We asume $\lim\limits_{n\to\infty}\alpha(n)=0$ because of the same reasons as in (**2.**).

2 Main result

We first prove four lemmas concerning any operator L which is satisfying the following two conditions:

(2.1) $\quad L$ *is linear and positive operator*;

(2.2) $\quad L(1,x)=1\ ,\ L(t,x)=x;$

As a corollary from (2.1) and (2.1) we obtain the following property

(2.3) $\quad f\leq Lf$ for convex function f.

Lemma 2.1. *For every function $f\in C_0(w)[0,1]$ we have $\|wL(f)\|\leq\|wf\|$, i.e. the norm of the operator is 1.*

Proof. Let we mention that function $(w)^{-1}$ is concave and then from (2.3)) we have $(w)^{-1}\geq L\left((w)^{-1}\right)$. The last one, (2.1) and (2.2) give

$$\begin{aligned}\|wL(f)\| &= \|wL\left(wf(w)^{-1}\right)\| \\ &\leq \|wf\|\,\|wL\left((w)^{-1}\right)\| \\ &\leq \|wf\|\,\|w(w)^{-1}\| = \|wf\|.\end{aligned}$$

$\square$

We define

$$K_y(x) \stackrel{\text{def}}{=} \begin{cases} y(x-1) & 0 \le y \le x \le 1; \\ x(y-1) & 0 \le x \le y \le 1, \end{cases}$$

Lemma 2.2. *For every $f \in W^2(w\varphi)$*

$$L(f,x) - f(x) = \int_0^1 \left(L(K_y,x) - K_y(x)\right) f''(y)dy.$$

The above statement is Lemma 3.1 from [3] .

We define $\quad f_w(x) = xf_0(x) + (1-x)f_1(x) \quad$ where
$$f_0(x) = -\int_x^1 \frac{dy}{y^{1+\gamma_0}(1-y)^{\gamma_1}} \qquad \text{and} \qquad f_1(x) = -\int_0^x \frac{dy}{y^{\gamma_0}(1-y)^{1+\gamma_1}}.$$

Lemma 2.3. *Let $f \in W^2(w\varphi)$, then we have*

$$\|w(Lf - f)\| \le \|w\varphi f''\| \, \|w(Lf_w - f_w\|.$$

Proof. The function $K_y(x)$ is convex and nonpositive. Then from conditions 2.1 and 2.3 it follows that $L(K_y,x) - K_y(x) \ge 0$.

From Lemma 2.2 we have

$$L(f,x) - f(x) = \int_0^1 \frac{L(K_y,x) - K_y(x)}{\varphi(y)} f''(y)\varphi(y)dy.$$

Taking a norm in the above equality we obtain

$$\begin{aligned} \|w(Lf - f)\| &= \left\| w \int_0^1 \frac{L(K_y) - K_y}{w(y)\varphi(y)} w(y) f''(y)\varphi(y)dy \right\| \\ &\le \|w\varphi f''\| \max_{x\in[0,1]} \left| w(x) \left(L \left(\int_0^1 \frac{K_y(x)}{w(y)\varphi(y)} dy, x \right) - \int_0^1 \frac{K_y(x)}{w(y)\varphi(y)} dy \right) \right|. \qquad (2.4) \end{aligned}$$

In the right hand side of the above inequality we have the function

$$
\begin{aligned}
(2.5)\qquad & \int_0^1 \frac{K_y(x)}{w(y)\varphi(y)}dy = \int_0^x \frac{y(x-1)}{y^{1+\gamma_0}(1-y)^{1+\gamma_1}}dy + \int_x^1 \frac{x(y-1)}{y^{1+\gamma_0}(1-y)^{1+\gamma_1}}dy \\
& = -(1-x)\int_0^x \frac{dy}{y^{\gamma_0}(1-y)^{1+\gamma_1}}dy - x\int_x^1 \frac{dy}{y^{1+\gamma_0}(1-y)^{\gamma_1}}dy \\
& = xf_0(x) + (1-x)f_1(x) \\
& = f_w(x).
\end{aligned}
$$

Replacing the result of 2.5 in 2.4 we obtain

$$
\begin{aligned}
\|w(Lf-f)\| & \le \|w\varphi f^{''}\| \max_{x\in[0,1]} |w(x)\left(L(f_w,x)-f_w(x)\right)| \\
& = \|w\varphi f^{''}\|\ \|w(Lf_w-f_w)\|.
\end{aligned}
$$

□

Lemma 2.4.

$$\|w(Lf_w - f_w)\| \le \|\varphi^{-1}(\cdot)L\left((t-\cdot)^2,\cdot\right)\|.$$

Proof. From the definition of f_w, 2.1 and 2.2 we have

$$
\begin{aligned}
(2.6)\qquad 0 & \le L(f_w,x) - f_w(x) \\
& = L\left(tf_1(t) + (1-t)f_0(t), x\right) - L(1-t,x)f_0(x) - L(t,x)f_1(x) \\
& = L\left((1-t)\left(f_0(t)-f_0(x)\right), x\right) + L\left(t\left(f_1(t)-f_1(x)\right), x\right).
\end{aligned}
$$

Expanding for $i = 0$, 1 functions $f_i(x+t-x)$ by Taylor's formula:

$$
\begin{aligned}
f_0(t) &= f_0(x) - \frac{t-x}{x^{\gamma_0}(1-x)^{1+\gamma_1}} + \int_x^t (t-u)f_0^{''}(u)du\,; \\
f_1(t) &= f_1(x) + \frac{t-x}{x^{1+\gamma_0}(1-x)^{\gamma_1}} + \int_x^t (t-u)f_1^{''}(u)du
\end{aligned}
$$

and using (from definitions of functions) that $f_0^{''}(u) < 0$ and $f_1^{''}(u) < 0$ we obtain

$$(1-t)\left(f_0(t)-f_0(x)\right) \le -\frac{(1-t)(t-x)}{x^{\gamma_0}(1-x)^{1+\gamma_1}}\,; \tag{2.7}$$

$$t\left(f_1(t)-f_1(x)\right) \le \frac{t(t-x)}{x^{1+\gamma_0}(1-x)^{\gamma_1}}. \tag{2.8}$$

Applying the results of 2.7 and 2.8 in 2.6 we have

$$\begin{aligned}
&0 \le w(x)\left(L(f_w,x) - f_w(x)\right) \\
&\le w(x)L\left(-\frac{(1-t)(t-x)}{x^{\gamma_0}(1-x)^{1+\gamma_1}} + \frac{t(t-x)}{x^{1+\gamma_0}(1-x)^{\gamma_1}}, x\right) \\
&= \varphi^{-1}(x)L\left((t-x)^2, x\right).
\end{aligned}$$

Taking a norm in the above inequality we prove Lemma 2.4. $\square$

Recapitulating results from above four lemmas we obtain

Theorem 2.1. *(Jackson-type inequality). Let L satisfies conditions 2.1 and 2.2. Then for every function $f \in W^2(w\varphi)$ we have*

$$\|w(Lf - f)\| \le \|w\varphi f''\| \ \|\varphi^{-1}(\cdot)L\left((t-\cdot)^2, \cdot\right)\|.$$

Let we mention that 1.2 and 1.3 are the properties 2.1 and 2.2 for operators $\tilde{B}_n$. From 1.3 and 1.4 it follows that

$$\frac{1}{\varphi(x)}\tilde{B}_n((t-x)^2, x) = \frac{\tilde{B}_n(t^2,x) - x^2}{\varphi(x)} = \alpha(n).$$

Above result and Theorem 2.1 give

Theorem 2.2. *For every function $f \in W^2(w\varphi)$ we have*

$$\|w(\tilde{B}_n f - f)\| \le \alpha(n)\|w\varphi f''\|.$$

The Theorem 2.2 we use in the proof of Theorem 1.1.

Proof of Theorem 1.1. Let g is an arbitrary function in $W^2(w\varphi)$. Then

$$\|w(\tilde{B}_n f - f)\| \le \|w(\tilde{B}_n f - \tilde{B}_n g)\| + \|w(\tilde{B}_n g - g)\| + \|w(g - f)\|.$$

From Lemma 2.1 and Theorem 2.2 we get

$$\|w(\tilde{B}_n f - f)\| \le 2\|w(f-g)\| + \alpha(n)\|w\varphi g''\| \le 2\left(\|w(f-g)\| + \frac{\alpha(n)}{2}\|w\varphi g''\|\right).$$

Taking an infimum on all $g \in W^2(w\varphi)$ in the above inequality we prove Theorem 1.1.

References

[1] T. N. T. GOODMAN AND A. SHARMA, A Bernstein-type Operator on the Simplex. *Mathematica Balcanica (new series) 5,2,* (1991), 129-145.

[2] T. N. T. GOODMAN AND A. SHARMA, A Modified Bernstein-Shoenberg Operator. *Proc. of Conference on Constructive Theory of Functions, Varna'87, Publ. House of Bulg. Acad. of Sci., Sofia,* (1987), 166-173.

[3] P. E. PARVANOV AND B. D. POPOV, The Limit Case of Bernstein's Operators with Jacobi-weights. *Mathematica Balcanica (new series), Vol.8, Fask.2-3,* (1994), 165-177.

[4] G. G. LORENTZ, Bernstein Polinonials. *Mathematical Expositions, 8, University of Toronto Press,* (1953).

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