WEIGHTED APPROXIMATION
BY KANTOROVICH TYPE MODIFICATION
OF MEYER-KÖNIG AND ZELLER OPERATOR

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We investigate the weighted approximation of functions in $L_p$-norm by Kantorovich modifications of the classical Meyer-König and Zeller operator, with weights of type $(1 - x)^\alpha$, $\alpha \in \mathbb{R}$. By defining an appropriate K-functional we prove direct theorems for them.

Keywords: Meyer-König and Zeller operator, K-functional, direct theorem, moduli of smoothness.


1. INTRODUCTION

In order to approximate unbounded functions in uniform norm in $[0,1)$, Meyer-König and Zeller (see [15]) introduced a new operator by the formula

$$M_n(f; x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n + k}\right), \quad (1.1)$$

where

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1 - x)^{n+1}. \quad (1.2)$$

But this operator cannot be used to approximate functions in $L_p$-norm because it is not bounded operator in $L_p$. Some kind of modification is needed. In this paper

we investigate the weighted approximation of functions in $L^p$-norm by Kantorovich modifications of the classical Meyer-König and Zeller (MKZ) operator.

In 1930, Kantorovich [13] suggested a modification of the classical Bernstein operator, replacing the function values by mean values. Analogously, Totik [16] introduced Kantorovich type modification of MKZ operator:

$$\tilde{M}_n^*(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(n + k)(n + k + 1)}{n} \int_{\frac{k}{n+k+1}}^{\frac{k+1}{n+k+2}} f(u) \, du,$$

and proved direct and converse theorems of weak type in terminology of Ditzian and Ivanov [4] for it. Although this definition looks as the most natural one, the operator $\tilde{M}_n^*$ is not a contraction, hence it is not very suitable for approximating functions in $L^p$-norm for $p < \infty$.

In [14] Müller defined a Kantorovich modification of MKZ operator in a slightly different way, so that the resulting operator is a contraction:

$$\tilde{M}_n(f;x) = \tilde{M}_n f(x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(n + k + 1)(n + k + 2)}{n + 1} \int_{\frac{k}{n+k+1}}^{\frac{k+1}{n+k+2}} f(u) \, du. \quad (1.3)$$

Recently, in [11] by introducing an appropriate K-functional the first author proved a direct theorem for the operators $\tilde{M}_n(f;x)$. Our goal in this paper is to extend this result for the case of weighted approximation of functions in $L^p$-norm by $\tilde{M}_n(f;x)$ operator.

Let us introduce some notations. For the sake of simplicity and brevity of our presentation we set

$$\gamma_{n,k} = \frac{(n + k + 1)(n + k + 2)}{n + 1}, \quad \Delta_{n,k} = \left[ \frac{k}{n + k + 1}, \frac{k + 1}{n + k + 2} \right]. \quad (1.4)$$

Then, the Kantorovich modification of MKZ operator (1.3) takes the form

$$\tilde{M}_n(f;x) = \sum_{k=0}^{\infty} \gamma_{n,k} m_{n,k}(x) \int_{\Delta_{n,k}} f(u) \, du.$$

The weights under consideration in our survey are

$$w(x) = (1 - x)^\alpha, \quad \alpha \in \mathbb{R}. \quad (1.5)$$

By $\varphi(x) = x(1 - x)^2$ we denote the weight which is naturally related to the second derivative of MKZ operator. The usual first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^k g(x) = g^{(k)}(x)$ for every $k \in \mathbb{N}$.

We define a differential operator $\tilde{D}$ by the formula

$$\tilde{D} = \frac{d}{dx} \left( \varphi(x) \frac{d}{dx} \right) = D \varphi D.$$
We set

\[ L_p(w) = \{ f : w f \in L_p(0,1) \}, \]

\[ W_p(w) = \left\{ \begin{array}{l} \{ f : f, Df \in AC_{loc}(0,1), wDf \in L_p[0,1], \lim_{x \to 0^+} \varphi(x)Df(x) = 0 \}, \alpha < 0, \\ \{ f : f, Df \in AC_{loc}(0,1), wDf \in L_p[0,1], \lim_{x \to 0^+} \varphi(x)Df(x) = 0 \}, \alpha \geq 0, \end{array} \right. \]

Also, we define a K-functional \( \tilde{K}_w(f,t)_p \) for \( t > 0 \) by

\[ \tilde{K}_w(f,t)_p = \inf \|w(f-g)\|_p + t\|wDg\|_p : f-g \in L_p(w), g \in W_p(w). \]  

(1.6)

Our main result is the following theorem.

**Theorem 1.** For \( 1 \leq p \leq \infty \), \( w \) defined by (1.5), \( \tilde{M}_n \) defined by (1.3), and the K-functional given by (1.6) there exists a positive constant \( C \) such that for every \( n > |\alpha|, n \in \mathbb{N} \), and for all functions \( f \in L_p(w) + W_p(w) \) there holds

\[ \|w(M_n f - f)\|_p \leq C \tilde{K}_w(f, \frac{1}{n})_p. \]  

(1.7)

**Remark 1.** Converse theorem remains an open problem even for the non-weighted case, i.e., for \( w(x) = 1 \) in (1.5).

Problems on characterization of weighted K-functionals by moduli of smoothness were considered by Draganov and Ivanov in [6, 7, 9]. Particularly, they characterized the K-functional

\[ K_w(f,t)_p = \inf \{ \|w(f-g)\|_p + t\|wD^2g\|_p : g, Dg \in AC_{loc}(0,1), f-g, \varphi D^2g \in L_p(w) \}. \]  

(1.8)

In this paper we also show that the same moduli of smoothness can be used for computing the K-functional \( K_w(f,t)_p \). So, we prove the next statement.

**Theorem 2.** For \( 1 < p < \infty \) and \( w \), \( \tilde{K}_w(f,t)_p \), \( K_w(f,t)_p \), defined by (1.5), (1.6) and (1.8), respectively, there exists a positive constant \( C \) such that for all \( f \in L_p(w) + W_p(w) \) there holds

\[ \tilde{K}_w(f,t)_p \leq C (K_w(f,t)_p + tE_0(f)), \]  

where \( E_0(f) = \inf_{c \in \mathbb{R}} \|w(f-c)\|_p \) is the best weighted approximation to \( f \) by a constant.

**Remark 2.** For \( p = 1 \) and \( p = \infty \) new moduli are needed. Also, a problem on characterization of the K-functional \( \tilde{K}_w(f,t)_p \) arises, but it is not the subject of our survey here.
Henceforth, the constant \( C \) will always be an absolute positive constant, which means it does not depend on \( f \) and \( n \). Also, it may be different on each occurrence. The relation \( \theta_1(f, t) \sim \theta_2(f, t) \) means that there exists a constant \( c \geq 1 \), independent of \( f \) and \( t \), such that

\[
c^{-1}\theta_1(f, t) \leq \theta_2(f, t) \leq c\theta_1(f, t).
\]

2. AUXILIARY RESULTS

In this section we present some properties of the operators \( M_n, \tilde{M}_n \), basis functions \( m_{n,k} \) (see [1, 10, 12]), and prove auxiliary lemmas that we need further.

The operators \( M_n \) and \( \tilde{M}_n \) are linear positive operators with

\[
\|M_n f\|_{\infty} \leq \|f\|_{\infty} \quad \text{and} \quad \|\tilde{M}_n\|_1 = 1.
\]

Moreover,

\[
\|\tilde{M}_n\|_p \leq 1, \quad 1 \leq p \leq \infty, \quad (2.1)
\]

\[
M_n(1; x) = 1, \quad M_n(t - x; x) = 0, \quad (2.2)
\]

\[
\tilde{M}_n(1; x) = 1. \quad (2.3)
\]

A direct integration yields the identity:

\[
\int_0^1 m_{n,k}(x)dx = \frac{1}{\gamma_{n,k}}. \quad (2.4)
\]

We shall need the next three properties of the functions \( \{m_{n,k}\}_{k=0}^{\infty} \), defined by (1.2) (for proofs, see e.g., [11]).

**Lemma 1.** If \( n \in \mathbb{N} \), then

\[
\frac{1}{1 - x} = \frac{1}{n + 1} \sum_{k=0}^{\infty} (n + k + 1)m_{n,k}(x), \quad x \in [0, 1). \quad (2.5)
\]

**Lemma 2.** If \( n \in \mathbb{N} \), then

\[
\sum_{k=1}^{n} \frac{(1 - x)^k}{k} = \sum_{k=0}^{\infty} m_{n,k}(x) \sum_{j=1}^{n} \frac{1}{k + j}, \quad x \in [0, 1). \quad (2.6)
\]

**Lemma 3.** There exists an absolute constant \( C \) such that for every \( n \in \mathbb{N} \) the following inequality holds true:

\[
\left| \ln(1 - x) + \sum_{k=0}^{\infty} m_{n,k}(x) \sum_{j=1}^{k+1} \frac{1}{n + j} \right| \leq \frac{C}{n}, \quad x \in [0, 1). \quad (2.7)
\]
In [16, Lemma 3] Totik proved that for $1 \leq p < \infty$,

$$
\| (1 - x) Df(x) \|_p \leq C(\| f \|_p + \| \varphi D^2 f \|_p).
$$

(2.8)

In order to prove our main results we need a few additional lemmas.

**Lemma 4.** For every integer $\nu$ there exists a constant $C = C(\nu)$, such that

$$
\sum_{k=0}^{\infty} \left(1 - \frac{k}{n + k + 1}\right)^\nu m_{n,k}(x) \leq C(1 - x)^\nu, \quad x \in [0,1),
$$

(2.9)

for all $n > |\nu|$, $n \in \mathbb{N}$.

**Proof.** We have

$$
\sum_{k=0}^{\infty} \left(1 - \frac{k}{n + k + 1}\right)^\nu m_{n,k}(x)
= \sum_{k=0}^{\infty} \left(\frac{n + 1}{n + k + 1}\right)^\nu \left(\frac{n + k}{k}\right)^x (1 - x)^{n+1}
= (1 - x)^\nu \sum_{k=0}^{\infty} \frac{(n + 1)^\nu (n + k - \nu + 1) \cdots (n + k)}{(n - \nu + 1) \cdots n (n + k + 1)^\nu} m_{n - \nu,k}(x)
\leq (1 - x)^\nu \sum_{k=0}^{\infty} C(\nu) m_{n - \nu,k}(x)
= C(\nu)(1 - x)^\nu.
$$

□

**Lemma 5.** For every $\alpha \in \mathbb{R}$ there exists a constant $C = C(\alpha)$, such that the following inequality is satisfied:

$$
\sum_{k=0}^{\infty} \left(1 - \frac{k}{n + k + 1}\right)^\alpha m_{n,k}(x) \leq C(1 - x)^\alpha, \quad x \in [0,1),
$$

(2.10)

for all $n > |\alpha|$, $n \in \mathbb{N}$.

**Proof.** Let $\nu$ be the smallest positive integer such that $\nu \geq |\alpha|$. Then, by Hölder’s inequality it follows that

$$
\sum_{k=0}^{\infty} \left(1 - \frac{k}{n + k + 1}\right)^\alpha m_{n,k}(x)
\leq \left( \sum_{k=0}^{\infty} \left(1 - \frac{k}{n + k + 1}\right)^{\nu \text{sign}(\alpha)} m_{n,k}(x) \right)^{\alpha/\nu} \left( \sum_{k=0}^{\infty} m_{n,k}(x) \right)^{1 - |\alpha|/\nu}.
$$
Applying Lemma 4 we obtain
\[
\left(\sum_{k=0}^{\infty} \left(1 - \frac{k}{n+k+1}\right)^{\nu \sign(\alpha)} m_{n,k}(x)\right)^{|\alpha|/\nu} \leq (C(1-x)^{\nu \sign(\alpha)})^{|\alpha|/\nu} = C(\alpha)(1-x)^{\alpha}.
\]
Therefore,
\[
\sum_{k=0}^{\infty} \left(1 - \frac{k}{n+k+1}\right)^{\alpha} m_{n,k}(x) \leq C(\alpha)(1-x)^{\alpha}
\]
and the lemma is proved. \(\square\)

The next lemma is a weighted variant of (2.1).

**Lemma 6.** Let \(1 \leq p \leq \infty\) and \(\alpha \in \mathbb{R}\). Then, there exists an absolute constant \(C\) such that for all \(n > |\alpha|, n \in \mathbb{N}\), and \(f \in L_p(w)\), we have
\[
\|w \tilde{M}_nf\|_p \leq C \|wf\|_p. \tag{2.11}
\]

**Proof.** First we prove (2.11) for \(p = 1\) and \(p = \infty\). Then, by applying Riesz-Thorin theorem we obtain the estimation for every \(1 < p < \infty\).

**The case** \(p = 1\). We have
\[
\|w \tilde{M}_nf\|_1 = \int_0^1 w(x) \left| \sum_{k=0}^{\infty} \gamma_{n,k} m_{n,k}(x) \int_{\Delta_{n,k}} f(t) \ dt \right| \ dx
\]
\[
\leq \int_0^1 w(x) \left[ \sum_{k=0}^{\infty} \gamma_{n,k} m_{n,k}(x) \int_{\Delta_{n,k}} \frac{|(wf)(t)|}{w(t)} \ dt \right] \ dx
\]
\[
\leq C \int_0^1 \left[ \sum_{k=0}^{\infty} \frac{w(x)}{w\left(\frac{k}{n+k+1}\right)} m_{n,k}(x) \int_{\Delta_{n,k}} |(wf)(t)| \ dt \right] \ dx
\]
\[
= C \int_0^1 \sum_{k=0}^{\infty} \left(1 - \frac{x}{1 - \frac{k}{n+k+1}}\right)^{\alpha} a_{n,k} m_{n,k}(x) \ dx,
\]
where we set
\[
a_{n,k} = \gamma_{n,k} \int_{\Delta_{n,k}} |(wf)(t)| \ dt.
\]

Let \(\nu = \lceil |\alpha| \rceil\) be the smallest positive integer such that \(\nu \geq |\alpha|\). Applying Hölder’s inequality twice we obtain
\[
\sum_{k=0}^{\infty} \left(\frac{1 - x}{1 - \frac{k}{n+k+1}}\right)^{\alpha} a_{n,k} m_{n,k}(x)
\]
\[
\leq \left[ \sum_{k=0}^{\infty} \left(\frac{1 - x}{1 - \frac{k}{n+k+1}}\right)^{\nu \sign(\alpha)} a_{n,k} m_{n,k}(x) \right]^{\nu/\nu} \left[ \sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x) \right]^{1-\nu/\nu},
\]

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thus
\[ ||w\hat{M}_n f||_1 \leq C \left( \sum_{k=0}^{\infty} \left( \frac{1 - x}{1 - \frac{k}{n+k+1}} \right)^{\nu \text{ sign}(\alpha)} a_{n,k} m_{n,k}(x) \right)^{\frac{\alpha}{\nu}} \]
\[ \times \left( \sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x) \right)^{1 - \frac{\alpha}{\nu}}. \]  \hspace{1cm} (2.12)

Now, we estimate the first nonconstant multiplier in the right-hand side of inequality (2.12). Let \( \ell = \nu \text{ sign}(\alpha) \). For every integer number \( \ell \) we have
\[ \left( \frac{1 - x}{1 - \frac{k}{n+k+1}} \right)^{\ell} m_{n,k}(x) = \frac{(n+k+1)^{\ell} (n+1) \cdots (n+\ell)}{(n+k+1) \cdots (n+k+\ell)} m_{n+\ell,k}(x) \]
\[ \leq C(\ell) m_{n+\ell,k}(x), \]
hence
\[ \sum_{k=0}^{\infty} \left( \frac{1 - x}{1 - \frac{k}{n+k+1}} \right)^{\ell} a_{n,k} m_{n,k}(x) \leq C(\ell) \sum_{k=0}^{\infty} a_{n,k} m_{n+\ell,k}(x). \]

Then, by (2.4),
\[ \left\| \sum_{k=0}^{\infty} \left( \frac{1 - x}{1 - \frac{k}{n+k+1}} \right)^{\ell} a_{n,k} m_{n,k}(x) \right\|_1 \leq C \left\| \sum_{k=0}^{\infty} a_{n,k} m_{n+\ell,k}(x) \right\|_1 \]
\[ \leq C \sum_{k=0}^{\infty} a_{n,k} \| m_{n+\ell,k}(x) \|_1 = C \sum_{k=0}^{\infty} \frac{\gamma_{n,k}}{\gamma_{n+\ell,k}} \int_{\Delta_{n,k}} |(wf)(t)| dt \]
\[ \leq C \sum_{k=0}^{\infty} \int_{\Delta_{n,k}} |(wf)(t)| dt = C \|wf\|_1. \]

Since \( \sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x) = \hat{M}_n(wf; x) \) and \( \|\hat{M}_n(wf)\|_1 \leq \|wf\|_1 \) by (2.1), then for the last multiplier in the right-hand side of (2.12) we obtain the inequality
\[ \| \sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x) \|_1 \leq \|wf\|_1. \]  Therefore,
\[ ||w\hat{M}_n f||_1 \leq C ||wf||_1^{\frac{\alpha}{\nu}} ||wf||_1^{1 - \frac{\alpha}{\nu}} = C ||wf||_1 \]
and the proof of the estimate (2.11) for \( p = 1 \) is complete.
The case $p = \infty$. We obtain
\[
\left| \frac{w(x)}{\Delta_{n,k}} \sum_{k=0}^{\infty} \Gamma_{n,k} m_{n,k}(x) \int_{\Delta_{n,k}} f(t) \, dt \right| \leq w(x) \sum_{k=0}^{\infty} \Gamma_{n,k} m_{n,k}(x) \int_{\Delta_{n,k}} \frac{|w f(t)|}{w(t)} \, dt \\
\leq C w(x) \sum_{k=0}^{\infty} \frac{\Gamma_{n,k} m_{n,k}(x)}{w(\frac{k}{n+k+1})} \int_{\Delta_{n,k}} |w f(t)| \, dt \\
\leq C w(x) \sum_{k=0}^{\infty} \frac{m_{n,k}(x)}{w(\frac{k}{n+k+1})} \left\| w f \right\|_{\infty} \\
= C w(x) \left\| w f \right\|_{\infty} \sum_{k=0}^{\infty} \left( 1 - \frac{k}{n+k+1} \right)^{-\alpha} m_{n,k}(x).
\]

Now, by Lemma 5 we have
\[
\sum_{k=0}^{\infty} \left( 1 - \frac{k}{n+k+1} \right)^{-\alpha} m_{n,k}(x) \leq C (1 - x)^{-\alpha}.
\]

Hence,
\[
\left\| w \tilde{M}_n f \right\|_{\infty} \leq C w(x) \left\| w f \right\|_{\infty} (1 - x)^{-\alpha} = C \left\| w f \right\|_{\infty},
\]
which proves (2.11) in the case $p = \infty$.

Finally, the inequality (2.11) follows for all $1 \leq p \leq \infty$ by the Riesz-Thorin interpolation theorem. □

The crucial result in our investigation is the following Jackson type inequality.

**Lemma 7.** Let $1 \leq p \leq \infty$ and $\alpha \in \mathbb{R}$. Then there exists an absolute constant $C$, such that for all $n > \|\alpha\|$, $n \in \mathbb{N}$, and $f \in W^p_w$, the following estimate holds true:
\[
\left\| w \tilde{M}_n f - f \right\|_p \leq \frac{C}{n} \left\| w \tilde{D} f \right\|_p.
\] (2.13)

(Let us note that the lemma implies that $\tilde{M}_n f - f \in L^p_w$ for $f \in W^p_w$.)

**Proof.** Let us set
\[
\phi(x) = \ln \frac{x}{1-x} + \frac{1}{1-x}, \quad x \in (0,1),
\]
with $\phi'(x) = \frac{1}{x(1-x)} = \frac{1}{\varphi(x)} > 0$, i.e., $\phi(x)$ is an increasing function. Then we have
\[
f(t) = f(x) + \varphi(x)[\phi(t) - \phi(x)] Df(x) + \int_x^t [\phi(u) - \phi(t)] Df(u) \, du, \quad t \in (0,1).
\]
Applying the operator $\tilde{M}_n$ to both sides of the latter equality and multiplying by $w(x)$ we obtain
\[
w(x)(\tilde{M}_nf(x) - f(x)) = w(x)\varphi(x)Df(x)[\tilde{M}_n\phi(x) - \phi(x)]
+ w(x)\tilde{M}_n\left(\int_x^{(1)} [\phi(\cdot) - \phi(u)]\tilde{D}f(u) \, du; x\right). \tag{2.14}
\]

First we prove the lemma for $p = 1$ and $p = \infty$. Then we apply the Riesz-Thornin theorem to obtain (2.13) for every $1 < p < \infty$.

The case $p = 1$. In order to prove that
\[
\|w\varphi Df [\tilde{M}_n\phi - \phi]\|_1 \leq \frac{C}{n}\|w\tilde{D}f\|_1 \tag{2.15}
\]
for all weights (1.5), we shall make use of the estimate
\[
\|\tilde{M}_n\phi - \phi\|_1 \leq \frac{C}{n} \tag{2.16}
\]
(see [11, Proof of Theorem 1] for a complete proof).

Let $\alpha > 0$ be fixed. Then, for all $n > \alpha$ and $f \in W_1(w)$ we have
\[
\varphi(x)Df(x) = \int_0^x (\varphi Df)'(u) \, du = \int_0^x \tilde{D}f(u) \, du, \quad x \in (0, 1).
\]
Hence,
\[
|w(x)\varphi(x)Df(x)| \leq w(x)\int_0^x |\tilde{D}f(u)| \, du \leq \int_0^x |(w\tilde{D}f)(u)| \, du \leq \int_0^1 |(w\tilde{D}f)(u)| \, du,
\]
i.e.,
\[
|w(x)\varphi(x)Df(x)| \leq \|w\tilde{D}f\|_1, \quad x \in (0, 1).
\]
Thus,
\[
\|w\varphi Df [\tilde{M}_n\phi - \phi]\|_1 \leq \|w\tilde{D}f\|_1 \|\tilde{M}_n\phi - \phi\|_1
\]
and (2.15) follows from (2.16).

Similarly, let $\alpha < 0$ be fixed. Then, for all $n > -\alpha$ we have $-n < \alpha < 0$ and for $f \in W_1(w)$, we consecutively obtain
\[
\varphi(x)Df(x) = \int_x^1 (\varphi Df)'(u) \, du = \int_x^1 \tilde{D}f(u) \, du, \quad x \in (0, 1),
\]
\[
|w(x)\varphi(x)Df(x)| \leq w(x)\int_x^1 |\tilde{D}f(u)| \, du \leq \int_x^1 |(w\tilde{D}f)(u)| \, du \leq \int_0^1 |(w\tilde{D}f)(u)| \, du,
\]
i.e.,
\[
|w(x)\varphi(x)Df(x)| \leq \|w\tilde{D}f\|_1, \quad x \in (0, 1).
\]
Hence, (2.16) yields (2.15).

Therefore, for arbitrary \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( f \in W_1(w) \) the estimate (2.15) holds true for \( n > |\alpha| \). The case \( \alpha = 0 \) was considered by the first author in [11].

Now, we estimate the \( L_1 \)-norm of the second summand in the right-hand side of (2.14). More precisely, we will prove

\[
\left\| w(x)M_n \left( \int_x^{(\cdot)} [\phi(\cdot) - \phi(u)] \hat{D}f(u) \, du \right) \right\|_1 \leq \frac{C}{n} \| \hat{w} \hat{D}f \|_1. \tag{2.17}
\]

Having in mind (1.4), for \( x \in (0,1) \) we have

\[
\left| w(x)M_n \left( \int_x^{(\cdot)} [\phi(\cdot) - \phi(u)] \hat{D}f(u) \, du \right) \right| \\
\leq w(x) \sum_{k=0}^{\infty} \gamma_{n,k} m_{n,k}(x) \int_{\Delta_{n,k}} \left( \int_x^t [\phi(t) - \phi(u)] \frac{|(w \hat{D}f)(u)|}{w(u)} \, du \right) dt \\
\leq C w(x) \sum_{k=0}^{\infty} \gamma_{n,k} m_{n,k}(x) \\
\times \left( \frac{1}{w(x)} + \frac{1}{w(n+k+1)} \right) \int_{\Delta_{n,k}} \left( \int_x^t [\phi(t) - \phi(u)] |(w \hat{D}f)(u)| \, du \right) dt \\
\leq C \sum_{k=0}^{\infty} \left( \frac{w(x)}{w(n+k+1)} + 1 \right) b_{n,k} m_{n,k}(x),
\]

where

\[
b_{n,k} = \gamma_{n,k} \int_{\Delta_{n,k}} \left( \int_x^t [\phi(t) - \phi(u)] |(w \hat{D}f)(u)| \, du \right) dt.
\]

Let \( \nu \) be the smallest positive integer such that \( \nu \geq |\alpha| \). Applying twice Hölder's inequality we obtain

\[
\sum_{k=0}^{\infty} \frac{w(x)}{w(n+k+1)} b_{n,k} m_{n,k}(x) \leq \left[ \sum_{k=0}^{\infty} \left( \frac{w(x)}{w(n+k+1)} \right)^{\nu/|\alpha|} b_{n,k} m_{n,k}(x) \right]^{\nu/|\alpha|} \\
\times \left[ \sum_{k=0}^{\infty} b_{n,k} m_{n,k}(x) \right]^{-1-|\alpha|/\nu},
\]

thus

\[
\left\| w(x)M_n \left( \int_x^{(\cdot)} [\phi(\cdot) - \phi(u)] \hat{D}f(u) \, du \right) \right\|_1 \\
\leq C \left\| \sum_{k=0}^{\infty} \left( \frac{w(x)}{w(n+k+1)} \right)^{\nu/|\alpha|} b_{n,k} m_{n,k} \right\|_1^{\nu/|\alpha|} \left\| \sum_{k=0}^{\infty} b_{n,k} m_{n,k} \right\|_1^{-1-|\alpha|/\nu}. \tag{2.18}
\]

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For estimation of the last factor in (2.18) we apply the estimate from [11] (see Proof of Theorem 1, Case 1, therein), by simply replacing \( \tilde{D}f \) with \( w \tilde{D}f \). So, we obtain

\[
\left\| \sum_{k=0}^{\infty} b_{n,k} m_{n,k} \right\|_1 \leq \frac{C}{n} \| w \tilde{D}f \|_1. \tag{2.19}
\]

Next, we focus on the estimating of the other multiplier in (2.18). Clearly,

\[
\left( \frac{1-x}{n+1} \right)^{n+k+1} m_{n,k}(x) = \frac{(n+k+1)^\ell (n+1) \cdots (n+k+1) (n+1) \cdots (n+k+1) \cdots (n+1)^\ell}{(n+k+1)^{n+k+1}} m_{n+\ell,k}(x)
\]

\[
\leq C(\ell) m_{n+\ell,k}(x)
\]

\[
\leq C(\ell) \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} m_{n+\ell,k}(x).
\]

Observe that the constant \( C(\ell) \) depends only on \( \alpha \).

We shall make use of the following operator defined by

\[
\tilde{M}_{n,\alpha}(f; x) = \sum_{k=0}^{\infty} \gamma_{n+\ell,k} m_{n+\ell,k}(x) \int_{\Delta_n,k} f(u) du. \tag{2.20}
\]

Then,

\[
\sum_{k=0}^{\infty} \left( \frac{w(x)}{w(\frac{k}{n+k+1})} \right)^{n+k+1} b_{n,k} m_{n,k}(x) \leq C \tilde{M}_{n,\alpha} \left( \int_x^{(\cdot)} |\phi(\cdot) - \phi(u)||w \tilde{D}f(u)| du; x \right).
\tag{2.21}
\]

In order to estimate the \( L_1 \)-norm of the right-hand side in (2.21) we follow an approach applied, e.g., in [2, pp. 41–43]. The proof in our case is much more complicated, because the operator \( \tilde{M}_{n,\alpha} \) does not preserve the constant functions. More precisely, it has the properties

\[
\| \tilde{M}_{n,\alpha} \|_1 = 1, \quad \tilde{M}_{n,\alpha}(1; x) = \sum_{k=0}^{\infty} \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} m_{n+\ell,k}(x).
\]

Let us write the operator \( \tilde{M}_{n,\alpha} \) from (2.20) in the form

\[
\tilde{M}_{n,\alpha}(f; x) = \int_0^{1} K_n(x,t) f(t) dt,
\]

where $K_n(\cdot, \cdot)$ is the related kernel. Introducing the functions

$$
\phi_1(x) = \ln x, \quad \phi_2(x) = -\ln(1 - x), \quad \phi_3(x) = \frac{1}{1 - x},
$$

we have $\phi(x) = \phi_1(x) + \phi_2(x) + \phi_3(x)$ and for $j = 1, 2, 3$,

$$
\tilde{M}_{n, \alpha} \left( \int_x^x [\phi_j(\cdot) - \phi_j(u)][(|\tilde{D}f|)(u)] du; x \right)
\begin{aligned}
&= \int_0^x K_n(x, t) \int_x^t [\phi_j(t) - \phi_j(u)][(\tilde{D}f)(u)] du \, dt \\
&\quad + \int_x^1 K_n(x, t) \int_x^t [\phi_j(t) - \phi_j(u)][(\tilde{D}f)(u)] du \, dt.
\end{aligned}
$$

Then, by Fubini’s theorem we obtain:

$$
\left\| \tilde{M}_{n, \alpha} \int_x^x [\phi(\cdot) - \phi(u)][(\tilde{D}f)(u)] du \right\|_1
\begin{aligned}
&= \int_0^1 (\tilde{D}f)(u) \sum_{j=1}^3 \left( \int_u^1 \tilde{M}_{n, \alpha} \left( [\phi_j(u) - \phi_j(\cdot)]_+ ; x \right) dx \\
&\quad + \int_0^u \tilde{M}_{n, \alpha} \left( [\phi_j(\cdot) - \phi_j(u)]_+ ; x \right) dx \right) du. \quad (2.22)
\end{aligned}
$$

To estimate the right-hand side of (2.22) we need estimations for the expressions in the sum for each of the functions $\phi_j$, $j = 1, 2, 3$.

First, for $\phi_1$, using

$$
\int_0^1 \tilde{M}_{n, \alpha} \left( [\phi_1(u) - \phi_1(\cdot)]_+ ; x \right) dx = \| \tilde{M}_{n, \alpha} \left( [\phi_1(u) - \phi_1(\cdot)]_+ ; x \right) \|_1
\begin{aligned}
&\leq \| [\phi_1(u) - \phi_1(x)]_+ \|_1 \\
&= \int_0^u (\phi_1(u) - \phi_1(x)) \, dx,
\end{aligned}
$$

we have

$$
\int_u^1 \tilde{M}_{n, \alpha} \left( [\phi_1(u) - \phi_1(\cdot)]_+ ; x \right) dx + \int_0^u \tilde{M}_{n, \alpha} \left( [\phi_1(\cdot) - \phi_1(u)]_+ ; x \right) dx
\begin{aligned}
&= \int_0^1 \tilde{M}_{n, \alpha} \left( [\phi_1(u) - \phi_1(\cdot)]_+ ; x \right) dx - \int_0^u \tilde{M}_{n, \alpha} \left( [\phi_1(u) - \phi_1(\cdot)]_+ ; x \right) dx \\
&\quad + \int_0^u \tilde{M}_{n, \alpha} \left( [\phi_1(\cdot) - \phi_1(u)]_+ ; x \right) dx
\end{aligned}
$$
\[ \begin{align*}
&\leq \int_0^u (\phi_1(u) - \phi_1(x)) \, dx + \int_0^u \tilde{M}_{n,\alpha}([\phi_1(\cdot) - \phi_1(u)]_+ - [\phi_1(u) - \phi_1(\cdot)]_+ : x) \, dx \\
&= u\phi_1(u) - \int_0^u \phi_1(x) \, dx + \int_0^u \tilde{M}_{n,\alpha}(\phi_1 ; x) \, dx - \phi_1(u) \int_0^u \tilde{M}_{n,\alpha}(1 ; x) \, dx \\
&= \int_0^u (\tilde{M}_{n,\alpha}(\phi_1 ; x) - \phi_1(x)) \, dx - \phi_1(u) \int_0^u (\tilde{M}_{n,\alpha}(1 ; x) - 1) \, dx.
\end{align*} \] (2.23)

Analogously, for \( \phi_j \), \( j = 2, 3 \), we obtain
\[ \begin{align*}
&\int_1^1 \tilde{M}_{n,\alpha}([\phi_j(u) - \phi_j(\cdot)]_+ : x) \, dx + \int_1^u \tilde{M}_{n,\alpha}([\phi_j(\cdot) - \phi_j(u)]_+ ; x) \, dx \\
&\leq \int_1^1 (\tilde{M}_{n,\alpha}(\phi_j ; x) - \phi_j(x)) \, dx - \phi_j(u) \int_1^1 (\tilde{M}_{n,\alpha}(1 ; x) - 1) \, dx.
\end{align*} \] (2.24)

Since for \( x, u \in (0, 1) \),
\[ |\tilde{M}_{n,\alpha}(1 ; x) - 1| = \left| \sum_{k=0}^{\infty} \frac{\gamma_{n,k}^{+\ell,k}}{\gamma_{n,k}} m_{n,+\ell,k}(x) - 1 \right| \leq \frac{C}{n}, \]
\[ |u\phi_1(u)| \leq C, \quad |(1 - u)\phi_2(u)| \leq C, \quad |(1 - u)\phi_3(u)| \leq C, \]
then
\[ \begin{align*}
|\phi_1(u) \int_0^u (\tilde{M}_{n,\alpha}(1 ; x) - 1) \, dx| &\leq \frac{C}{n}, \\
|\phi_j(u) \int_1^1 (\tilde{M}_{n,\alpha}(1 ; x) - 1) \, dx| &\leq \frac{C}{n}, \quad j = 2, 3.
\end{align*} \] (2.25)

1. **Estimation of** \( \left| \int_0^u (\tilde{M}_{n,\alpha}(\phi_1 ; x) - \phi_1(x)) \, dx \right| \). **We have**
\[ \int_{\Delta_{n,k}} \phi_1(t) \, dt = \frac{k + 1}{n + k + 2} \ln \frac{k + 1}{n + k + 2} - \frac{k}{n + k + 1} \ln \frac{k}{n + k + 1} - \frac{1}{\gamma_{n,k}}, \]
and for \( x \in (0, 1) \),
\[ \phi_1(x) = -\sum_{k=1}^{n+\ell} \frac{(1 - x)^k}{k} - \sum_{k=n+\ell+1}^{\infty} \frac{(1 - x)^k}{k}. \]

By Lemma 2,
\[ \sum_{k=1}^{n+\ell} \frac{(1 - x)^k}{k} = \sum_{k=0}^{\infty} m_{n,\ell,k}(x) \sum_{i=1}^{n+\ell} \frac{1}{k+i}. \]
and therefore
\[
\left| \int_0^u (\tilde{M}_{n,\alpha}(\phi_1; x) - \phi_1(x)) \, dx \right|
\]
\[
= \left| \int_0^u \sum_{k=0}^\infty m_{n+\ell,k}(x) \left[ \gamma_{n+\ell,k} \int_{\Delta_{n,k}} \phi_1(t) \, dt + \sum_{i=1}^{n+\ell} \frac{1}{k+i} \right] \, dx + \int_0^u \sum_{k=n+\ell+1}^\infty \frac{(1-x)^k}{k} \, dx \right|
\]
\[
\leq \left| \int_0^u \sum_{k=0}^\infty m_{n+\ell,k}(x) \left[ \gamma_{n+\ell,k} \int_{\Delta_{n,k}} \phi_1(t) \, dt + \sum_{i=1}^{n+\ell} \frac{1}{k+i} \right] \, dx \right| + \frac{C}{n}.
\]

For \( k \geq 1 \),
\[
\ln \frac{k+1}{n+k+2} = -\ln \prod_{i=1}^{n+1} \frac{k+i+1}{k+i} = -\sum_{i=1}^{n+1} \ln \left( 1 + \frac{1}{k+i} \right)
\]
\[
= -\sum_{i=1}^{n+1} \left[ \frac{1}{k+i} - \frac{1}{2(k+i)^2} + \mathcal{O}\left( \frac{1}{(k+i)^3} \right) \right],
\]
and
\[
\sum_{i=1}^{n+1} \frac{1}{(k+i)^2} = \sum_{i=1}^{n+1} \left[ \frac{1}{(k+i)(k+i+1)} + \mathcal{O}\left( \frac{1}{(k+i)^3} \right) \right]
\]
\[
= \frac{n+1}{(k+1)(n+k+2)} + \sum_{i=1}^{n+1} \mathcal{O}\left( \frac{1}{(k+i)^3} \right),
\]
hence
\[
\ln \frac{k+1}{n+k+2} = -\sum_{i=1}^{n+1} \frac{1}{k+i} + \frac{n+1}{2(k+1)(n+k+2)} + \mathcal{O}\left( \frac{1}{k^2} \right).
\]
Since
\[
\frac{k+1}{n+k+2} \mathcal{O}\left( \frac{1}{k^2} \right) = \mathcal{O}\left( \frac{1}{k^2} \right),
\]
then
\[
\frac{k+1}{n+k+2} \ln \frac{k+1}{n+k+2} = -\frac{k+1}{n+k+2} \sum_{i=1}^{n+1} \frac{1}{k+i} + \frac{n+1}{2(n+k+2)^2} + \mathcal{O}\left( \frac{1}{k^2} \right).
\]
Similarly,
\[
\frac{k}{n+k+1} \ln \frac{k}{n+k+1} = -\frac{k}{n+k+1} \sum_{i=0}^{n} \frac{1}{k+i} + \frac{n+1}{2(n+k+1)^2} + \mathcal{O}\left( \frac{1}{k^2} \right).
\]

Therefore,

\[
\int_{\Delta_{n,k}} \phi_1(t) \, dt = \frac{k}{n + k + 1} \sum_{i=0}^{n} \frac{1}{k + i} - \frac{k + 1}{n + k + 2} \sum_{i=1}^{n+1} \frac{1}{k + i} - \frac{n + 1}{2} \left[ \frac{1}{(n + k + 1)^2} - \frac{1}{(n + k + 2)^2} \right] + O\left(\frac{1}{k^2}\right) - \frac{1}{\gamma_{n,k}} \sum_{i=0}^{n} \frac{1}{k + i} + O\left(\frac{1}{k^2}\right).
\]

Now, we have

\[
|M_{n,\alpha}(\phi_1; x) - \phi_1(x)| \leq m_{n,\ell,0}(x) \left|\ln(n + 2) + 1 - \sum_{i=1}^{n+\ell} \frac{1}{i}\right| + \sum_{k=1}^{\infty} m_{n,\ell,k}(x) \left|\frac{\gamma_{n,\ell,k}}{\gamma_{n,k}} \sum_{i=1}^{n} \frac{1}{k + i} - \sum_{i=1}^{n+\ell} \frac{1}{k + i}\right| + \frac{C}{n}.
\]

From

\[
\left|\ln(n + 2) + 1 - \sum_{i=1}^{n+\ell} \frac{1}{i}\right| \leq C, \quad \|m_{n,\ell,0}\|_1 \leq \frac{C}{n},
\]

it follows

\[
\left\|m_{n,\ell,0}(x) \left|\ln(n + 2) + 1 - \sum_{i=1}^{n+\ell} \frac{1}{i}\right|\right\|_1 \leq \frac{C}{n}.
\]

Moreover,

\[
\sum_{k=1}^{\infty} m_{n,\ell,k}(x) \left|\frac{\gamma_{n,\ell,k}}{\gamma_{n,k}} \sum_{i=1}^{n} \frac{1}{k + i} - \sum_{i=1}^{n+\ell} \frac{1}{k + i}\right| \leq \sum_{k=1}^{\infty} m_{n,\ell,k}(x) \left|\frac{\gamma_{n,\ell,k}}{\gamma_{n,k}} - 1\right| \sum_{i=1}^{n} \frac{1}{k + i} + \sum_{k=1}^{\infty} m_{n,\ell,k}(x) \sum_{i=n+1}^{n+\ell} \frac{1}{k + i}.
\]

Now, the inequalities

\[
\left|\frac{\gamma_{n,\ell,k}}{\gamma_{n,k}} - 1\right| \leq \frac{C}{n}, \quad \sum_{k=1}^{\infty} m_{n,\ell,k}(x) \sum_{i=n+1}^{n+\ell} \frac{1}{k + i} \leq \frac{C}{n} \sum_{k=1}^{\infty} m_{n,\ell,k}(x) \sum_{i=1}^{n} \frac{1}{k + i} + \frac{C}{n},
\]

yield

\[
\sum_{k=1}^{\infty} m_{n,\ell,k}(x) \left|\frac{\gamma_{n,\ell,k}}{\gamma_{n,k}} \sum_{i=1}^{n} \frac{1}{k + i} - \sum_{i=1}^{n+\ell} \frac{1}{k + i}\right| \leq \frac{C}{n} \sum_{k=1}^{\infty} m_{n,\ell,k}(x) \sum_{i=1}^{n} \frac{1}{k + i} + \frac{C}{n}.
\]
By Lemma 2 we obtain
\[
\sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n} \frac{1}{k+i} \leq \sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n+\ell} \frac{1}{k+i} \leq |\ln x|.
\]

Therefore,
\[
\left| \int_{0}^{u} \sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n} \frac{1}{k+i} \, dx \right| \leq \left| \int_{0}^{u} \ln x \, dx \right| \leq \left| \int_{0}^{1} \ln x \, dx \right| \leq C,
\]
and we conclude that
\[
\left| \int_{0}^{u} (\tilde{M}_{n,\alpha}(\phi_1; x) - \phi_1(x)) \, dx \right| \leq \frac{C}{n}.
\] (2.26)

2. Estimation of \( \left| \int_{u}^{1} (\tilde{M}_{n,\alpha}(\phi_2; x) - \phi_2(x)) \, dx \right| \). We have
\[
\int_{\Delta_{n,k}} \phi_2(t) \, dt = \frac{n+1}{n+k+2} \ln \frac{n+1}{n+k+2} - \frac{n+1}{n+k+1} \ln \frac{n+1}{n+k+1} + \frac{1}{\gamma_{n,k}},
\]
\[
\gamma_{n,k} \int_{\Delta_{n,k}} \phi_2(t) \, dt = 1 - (n+k+1) \ln \left( 1 + \frac{1}{n+k+1} \right) - \ln \frac{n+1}{n+k+1}
\]
\[
= \ln \frac{n+k+1}{n+1} + O \left( \frac{1}{n+k} \right),
\]
hence,
\[
M_{n,\alpha}(\phi_2; x) = \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \left[ \ln \frac{n+k+1}{n+1} + O \left( \frac{1}{n+k} \right) \right].
\]

Applying Lemma 3 we obtain
\[
\left| \phi_2(x) - \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \right| \leq \frac{C}{n},
\]
and then
\[
\left| \tilde{M}_{n,\alpha}(\phi_2; x) - \phi_2(x) \right| \leq \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \ln \frac{n+k+1}{n+1} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \right| + \frac{C}{n}.
\]
Taking into account that
\[
\ln \frac{n+k+1}{n+1} = \sum_{i=1}^{k} \ln \left( 1 + \frac{1}{n+i} \right) = \sum_{i=1}^{k} \frac{1}{n+i} + \sum_{i=1}^{k} O \left( \frac{1}{(n+i)^2} \right)
\]
\[
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and
\[ \sum_{i=1}^{k} \frac{1}{(n+i)^2} \leq \frac{C}{n}, \]
we estimate
\[ |\tilde{M}_{n,\alpha}(\phi_2; x) - \phi_2(x)| \leq \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \left| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \right| + \frac{C}{n} \]
\[ \leq \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \left| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \right| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \]
\[ + \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \left| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \right| + \frac{C}{n}. \]

Since
\[ \left| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \right| \leq \frac{C}{n}, \]
it follows that
\[ \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \left| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \right| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \leq \frac{C}{n} \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k} \frac{1}{n+i}. \]

Observe that
\[ \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \left| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \right| \leq \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{\ell} \frac{1}{n+i} \leq \frac{C}{n}. \]

We recall that \( \ell = \lceil |\alpha| \rceil \text{sign} (\alpha) \) and \( C = C(\alpha) \), i.e. \( C \) is an absolute constant for a fixed \( \alpha \). Then, by Lemma 3 we obtain
\[ \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \left| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \right| \sum_{i=1}^{k} \frac{1}{n+i} \]
\[ \leq \frac{C}{n^2} + \frac{C}{n} \ln(1-x) + \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} + \frac{C}{n} \ln(1-x) \]
\[ \leq \frac{C}{n^2} + \frac{C}{n^2} + \frac{C}{n} |\ln(1-x)|. \]

Therefore,
\[ \left| \int_{u}^{1} (\tilde{M}_{n,\alpha}(\phi_2; x) - \phi_2(x)) \, dx \right| \leq \frac{C}{n} \int_{0}^{1} (2 - \ln(1-x)) \, dx \leq \frac{C}{n}. \]  
(2.27)
3. Estimation of \( \int_u^1 (\tilde{M}_{n,\alpha}(\phi_3; x) - \phi_3(x)) \, dx \). The last estimation we need concerns the function \( \phi_3(x) = \frac{1}{1-x} \). We have
\[
\int_{\Delta_n,k} \phi_3(t) \, dt = \ln \left( 1 + \frac{1}{n+k+1} \right) = \frac{1}{n+k+1} + \mathcal{O} \left( \frac{1}{(n+k)^2} \right),
\]
\[
\gamma_{n,k} \int_{\Delta_n,k} \phi_3(t) \, dt = \frac{n+k+2}{n+1} + \mathcal{O} \left( \frac{1}{n} \right).
\]
By Lemma 1,
\[
\phi_3(x) = \frac{1}{n+\ell+1} \sum_{k=0}^{\infty} (n+\ell+k+1)m_{n+\ell,k}(x),
\]
hence
\[
|M_{n,\alpha}(\phi_3; x) - \phi_3(x)| \leq \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \frac{n+k+\ell+1}{n+\ell+1} \left( \frac{n+k+\ell+2}{n+k+1} - 1 \right) + \mathcal{O} \left( \frac{1}{n} \right)
\]
\[
= \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \frac{n+k+\ell+1}{n+\ell+1} \cdot \frac{\ell+1}{n+k+1} + \mathcal{O} \left( \frac{1}{n} \right) = \mathcal{O} \left( \frac{1}{n} \right).
\]
Then
\[
\left| \int_u^1 (\tilde{M}_{n,\alpha}(\phi_3; x) - \phi_3(x)) \, dx \right| \leq \frac{C}{n} \int_u^1 \, dx \leq \frac{C}{n}. \quad (2.28)
\]
Now, from inequalities (2.22)–(2.28) it follows that
\[
\left\| \tilde{M}_{n,\alpha} \int_x^{(\cdot)} \left[ \phi(\cdot) - \phi(u) \right] (w\tilde{D}f)(u) \, du \right\|_1 \leq \frac{C}{n}. \quad (2.29)
\]
The estimate (2.17) is a consequence of (2.18), (2.19), (2.21), and (2.29).

Finally, the estimate (2.13) for the case \( p = 1 \) follows from (2.14), (2.15) and (2.17).

The case \( p = \infty \).

We proceed similarly to the case \( p = 1 \): applying Holder’s inequality for the smallest integer \( \geq \alpha \), considering again the operator \( \tilde{M}_{n,\alpha} \) and using the following estimation
\[
\tilde{M}_{n,\alpha} \left( \int_x^{(\cdot)} \left[ \phi(\cdot) - \phi(u) \right] (w\tilde{D}f)(u) \, du; x \right)
\]
\[
\leq \|w\tilde{D}f\|_{\infty} \tilde{M}_{n,\alpha} \left( \int_x^{(\cdot)} \left[ \phi(\cdot) - \phi(u) \right] \, du; x \right)
\]
\[
\leq x |\tilde{M}_{n,\alpha}(\ln t; x) - \ln x| \|w\tilde{D}f\|_{\infty} + (1-x) |\tilde{M}_{n,\alpha}(\frac{1}{1-t}; x) - \frac{1}{1-x} \|w\tilde{D}f\|_{\infty} + x |\tilde{M}_{n,\alpha}(\ln(1-t); x) - \ln(1-x)| \|w\tilde{D}f\|_{\infty}.
\]
\[\square\]
For the proof of Theorem 2 we need a weighted variant of (2.8).
Lemma 8. Let $1 < p < \infty$. Then, for all functions $f \in L_p(w)$ such that $\varphi D^2 f \in L_p(w)$, there exists a constant $C$ such that the next inequality is true
\[
\|wD\varphi Df\|_p \leq C(\|wf\|_p + \|w\varphi D^2 f\|_p).
\]

Proof. The proof is analogous to the proof of [16, Lemma 3], using the obvious
\[
|D\varphi(x)| = |(1-x)(1-3x)| < 2(1-x), \quad 0 \leq x < 1,
\]
and $w(x) \sim w(1 - 2^{-k})$ for $x \in (1 - 2^{-k}, 1 - 2^{-k-1})$. \hfill \Box

3. PROOFS OF THEOREM 1 AND THEOREM 2

Proof of Theorem 1. We establish the direct inequality by means of a standard argument.

Let $1 \leq p \leq \infty$. For any $g \in W_p(w)$ such that $f - g \in L_p(w)$ we have, by virtue of (2.11) and Lemma 7,
\[
\|w(f - \tilde{M}_n f)\|_p \leq \|w(f - g)\|_p + \|w(g - \tilde{M}_n g)\|_p + \|w\tilde{M}_n(f - g)\|_p
\]
\[
\leq 2\|w(f - g)\|_p + \frac{C}{n} \|w\tilde{D} g\|_p
\]
\[
\leq C\left(\|w(f - g)\|_p + \frac{1}{n} \|w\tilde{D} g\|_p\right).
\]
Taking the infimum on $g$ we obtain the inequality (1.7) in the theorem. \hfill \Box

Proof of Theorem 2. For every $c \in \mathbb{R}$, by virtue of Lemma 8, we have
\[
\|wD\varphi Dg\|_p = \|wD\varphi D(g - c)\|_p
\]
\[
\leq C(\|w\varphi D^2 (g - c)\|_p + \|w(g - c)\|_p)
\]
\[
= C(\|w\varphi D^2 g\|_p + \|w(g - c)\|_p).
\]
Using the latter inequality and the obvious
\[
\|w\tilde{D} g\|_p \leq \|wD\varphi Dg\|_p + \|w\varphi D^2 g\|_p,
\]
we have for $t > 0$
\[
\|w(f - g)\|_p + t\|w\tilde{D} g\|_p
\]
\[
\leq \|w(f - g)\|_p + t\|wD\varphi Dg\|_p + t\|w\varphi D^2 g\|_p
\]
\[
= \|w(f - g)\|_p + Ct(\|w\varphi D^2 g\|_p + \|w(g - c)\|_p) + t\|w\varphi D^2 g\|_p
\]
\[
\leq C(\|w(f - g)\|_p + t\|w\varphi D^2 g\|_p) + Ct\|w(g - f + f - c)\|_p
\]
\[
\leq C(\|w(f - g)\|_p + t\|w\varphi D^2 g\|_p) + Ct\|w(g - f)\|_p + Ct\|w(f - c)\|_p
\]
\[
\leq C(\|w(f - g)\|_p + t\|w\varphi D^2 g\|_p + t\|w(f - c)\|_p).
\]

By taking infimum over all functions $g \in W_p(w)$ and all real constants $c$ we obtain the inequality

$$
\tilde{K}_w(f, t)_p \leq C \inf \{ \|w(f - g)\|_p + t \|w \varphi D^2 g\|_p : f - g \in L_p(w), g \in W_p(w) \}
+ CtE_0(f).
$$

To complete the proof in the case $\alpha \geq 0$, it remains to take into consideration that in the definition of $K_w(f, t)_p$ we can, equivalently, assume that $g$ is in $C^2$ in a neighbourhood of 0 if $f \in L_p(w)$ (see [3, p. 110]).

To complete the proof for $\alpha < 0$, we will show that if $g, Dg \in AC_{loc}(0, 1)$ and $wg, w \varphi D^2 g \in L_p[0, 1)$, then

$$
\lim_{x \to 1^-} \varphi(x)Dg(x) = 0.
$$

To this end, we first apply [5, Lemma 1] to get $(1 - x)^{\alpha + 1}Dg(x) \in L_p[1/2, 1)$.

Next, we use [8, Lemma 3.1(a)], transformed for a singularity at $x = 1$, with $G = \varphi Dg$ and $\gamma = \alpha - 1 < -1$ to derive

$$
\lim_{x \to 1^-} G(x) = \lim_{x \to 1^-} \varphi(x)Dg(x) = 0.
$$

\[\square\]

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4. REFERENCES


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