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WEIGHTED APPROXIMATION IN UNIFORM NORM BY MEYER-KÖNIG AND ZELLER OPERATORS

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The weighted approximation errors of Meyer-König and Zeller operator is characterized for weights of the form $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$, where $\gamma_0 \in [-1, 0], \gamma_1 \in \mathbb{R}$. Direct inequalities and strong converse inequalities of type A are proved in terms of the weighted K-functional.

Keywords: Meyer-König and Zeller operator, K-functional, Direct theorem, Strong converse theorem, Weighted approximation.

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1 Introduction and statement of the results

The classical Meyer-Konig and Zeller (MKZ) operator is defined for functions $f \in C[0, 1)$ by the formula

$$M_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x) \tag{1}$$

where

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}.$$

Right after their appearance, the MKZ operators became a subject of serious investigations. The reason for this is the fact, that they allow approximating of functions unbounded at the point 1 (which is different, comparing with Bernstein polynomials). But the values of the function are taken at the points $\frac{k}{n+k}$, which creates additional difficulties working with these operators.

In this paper we investigate the weighted approximation of functions by the classical variant of MKZ operator in uniform norm $\|.\|_{[0,1)}$, i.e. we want to characterize the weighted error of approximation $\sup_{x\in[0,1)} |w(x)f(x)|$, where

$$w(x) = x^{\gamma_0}(1-x)^{\gamma_1}. \tag{2}$$

are the Jacobi weights.

In the unweighted case ($w(x) = 1$) the direct theorem is proved in [4], and the strong converse inequality of type A (in terminology of [3]) is proved in [5]. Regarding the weighted case, the first results are obtained by Becker and Nessel in [2], where they proved the direct theorems for some symmetrical weights $w(x) = \varphi^{\alpha}(x)$ where $\varphi(x) = x(1-x)^2$ is the weight function which is naturally connected with the second derivative of MKZ operators.

In [10] Totik established, that for $0 < \alpha \leq 1$ and $\varphi(x) = x(1-x)^2$ the condition

$$\varphi^{\alpha}|\Delta_h^2(f,x)| \leq Kh^{2\alpha}$$

is equivalent to

$$M_n f - f = O\left(n^{-\alpha}\right).$$

In [9] the authors proved that for $0 \leq \lambda \leq 1$ and $0 < \alpha < 2$ the condition

$$|M_n f(x) - f(x)| = O\left(\left(\frac{\varphi^{(1-\lambda)/2}(x)}{\sqrt{n}}\right)^{\alpha}\right)$$

is equivalent to

$$\omega^2_{\varphi^{\lambda/2}}(f,t) = O(t^{\alpha}).$$

Here $\omega^2_{\varphi^{\lambda/2}}(f,t)$ are the modulus of Ditzian-Totik of second order

$$\omega^2_{\varphi^{\lambda/2}}(f,t) = \sup_{0<h\leq t} \sup_{x\pm h\varphi^{\lambda/2}(x)\in[0,1)} |\Delta^2_{h\varphi^{\lambda/2}(x)} f(x)|.$$

In [7] Holhoş proved the next direct inequality for weights $\gamma_0 = 0, \gamma_1 > 0$:

$$\|w(M_n f - f)\|_{[0,1)} \leq 2\omega\left(f(1-e^{-t})e^{-\gamma_1 t}, \frac{1}{\sqrt{n}}\right) + \frac{\gamma_1 C(\gamma_1)}{\sqrt{n}}\|wf\|_{[0,1)}.$$

In this paper we prove better results then all the results mention above. But before stating our main result, let us introduce some notations and definitions. The first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^2 g(x) = g''(x)$.

By $C[0,1)$ we denote the space of all continuous on $[0,1)$ functions. The functions from $C[0,1)$ are not expected to be continuous or bounded at 1. By $L_\infty[0,1)$ we denote the space of all Lebesgue measurable and essentially bounded in $[0,1)$ functions equipped with the uniform norm $\|\cdot\|_{[0,1)}$. For a weight function w we set

$$C(w)[0,1) = \{g \in C[0,1); \quad wg \in L_\infty[0,1)\},$$
$$W^2(w\varphi)[0,1) = \left\{g, Dg \in AC_{loc}(0,1) \ \& \ w\varphi D^2 g \in L_\infty[0,1)\right\},$$
$$W^3(w\varphi^{3/2})[0,1) = \left\{g, Dg, D^2 g \in AC_{loc}(0,1) \ \& \ w\varphi^{3/2} D^3 g \in L_\infty[0,1)\right\},$$

where $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,1)$.

The weighted approximation error $\|w(f-M_nf)\|_{[0,1)}$ will be compared with the K-functional between the weighted spaces $C(w)[0,1)$ and $W^2(w\varphi)[0,1)$, which for every

$$f\in C(w)[0,1)+W^2(w\varphi)[0,1)=\{f_1+f_2 : f_1\in C(w)[0,1), f_2\in W^2(w\varphi)[0,1)\}$$

and $t>0$ is defined by

$$K_w(f,t)_{[0,1)}=\inf_{g\in W^2(w\varphi),\ f-g\in C(w)}\left\{\|w(f-g)\|_{[0,1)}+t\|w\varphi D^2g\|_{[0,1)}\right\}. \qquad (3)$$

Our main result is the following theorem, establishing a full equivalence between the K-functional $K_w\left(f,\frac{1}{n}\right)_{[0,1)}$ and the weighted error $\|w(M_nf-f)\|_{[0,1)}$.

Theorem 1.1. *For w defined by (2), where $\gamma_0\in[-1,0],\gamma_1\in\mathbb{R}$, there exist positive constants C_1, C_2 and L such that for every natural $n\geq L$ and for all*

$$f\in C(w)[0,1)+W^2(w\varphi)[0,1)$$

there holds

$$C_1\|w(M_nf-f)\|_{[0,1)}\leq K_w\left(f,\frac{1}{n}\right)_{[0,1)}\leq C_2\|w(M_nf-f)\|_{[0,1)}. \qquad (4)$$

The proof is based on the method, used for the first time in [8]. Shortly, the idea is this: by making an appropriate transformation we go to Baskakov operators for which we have the needed estimations and go back by the inverse transformation.

2 A connection between Baskakov and MKZ operators

Following [8] we introduce a transformation T mapping functions defined on $[0,\infty)$ into functions defined on $[0,1)$. And we make the agreement that from now on we shall denote variables, functions and operators, defined in $[0,1)$ the usual way, and their analogs, defined in $[0,\infty)$, with tilde.

Now we give some notations and definitions.
The uniform norm on the interval $[0,\infty)$ we will denote $\|\cdot\|_{[0,\infty)}$ and we define the next function spaces.

$$C(\tilde{w})[0,\infty)=\{\tilde{g}\in C[0,\infty);\quad \tilde{w}\tilde{g}\in L_\infty[0,\infty)\},$$
$$W^2(\tilde{w}\tilde{\varphi})[0,\infty)=\left\{\tilde{g},\tilde{D}\tilde{g}\in AC_{loc}(0,\infty)\ \&\ \tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{g}\in L_\infty[0,\infty)\right\},$$
$$W^3(\tilde{w}\tilde{\varphi}^{3/2})[0,\infty)=\left\{\tilde{g},\tilde{D}\tilde{g},\tilde{D}^2\tilde{g}\in AC_{loc}(0,\infty)\ \&\ \tilde{w}\tilde{\varphi}^{3/2}\tilde{D}^3\tilde{g}\in L_\infty[0,\infty)\right\}.$$

The weighted error by Baskakov operators will be characterized by the next K-functional, defined for every function $\tilde{f} \in C(\tilde{w})[0,\infty) + W^2(\tilde{w}\tilde{\varphi})[0,\infty)$ and for every $t > 0$ by the formula

$$K_{\tilde{w}}(\tilde{f},t)_{[0,\infty)} = \inf \left\{ \|\tilde{w}(\tilde{f}-\tilde{g})\|_{[0,\infty)} + t \left\| \tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{g} \right\|_{[0,\infty)} \right\}, \tag{5}$$

where the infimum is taken over functions $\tilde{g} \in W^2(\tilde{w}\tilde{\varphi})[0,\infty)$ such that $\tilde{f} - \tilde{g} \in C(\tilde{w})[0,\infty)$.

We start with the change of variable $\sigma : [0,1) \to [0,\infty)$ (used for the first time by V.Totik in [10]) given by

$$\tilde{x} = \sigma(x) = \frac{x}{1-x}. \tag{6}$$

Then the inverse change of variable $\sigma^{-1} : [0,\infty) \to [0,1)$ is

$$x = \sigma^{-1}(\tilde{x}) = \frac{\tilde{x}}{1+\tilde{x}}.$$

The transformation operator T, transforming a function $\tilde{f}$ defined on $[0,\infty)$ to a function f defined on $[0,1)$ is defined by

$$f(x) = T(\tilde{f})(x) = \lambda(x)(\tilde{f} \circ \sigma)(x), \quad \lambda(x) = 1-x. \tag{7}$$

Then the inverse operator T^{-1}, transforming a function f defined on $[0,1)$ to a function $\tilde{f}$ defined on $[0,\infty)$ is

$$\tilde{f}(\tilde{x}) = T^{-1}(f)(\tilde{x}) = \frac{1}{(\lambda \circ \sigma^{-1})(\tilde{x})}(f \circ \sigma^{-1})(\tilde{x}).$$

We want to estimate the weighted error by MKZ, so we define a new transformation operator S by

$$w(x) = S(\tilde{w})(x) = \frac{1}{\lambda(x)}(\tilde{w} \circ \sigma)(x). \tag{8}$$

and its inverse S^{-1} is

$$\tilde{w}(\tilde{x}) = S^{-1}(w)(\tilde{x}) = (\lambda \circ \sigma^{-1})(\tilde{x})(w \circ \sigma^{-1})(\tilde{x}). \tag{9}$$

Obviously we have:

$$\begin{aligned} wf = S(\tilde{w})T(\tilde{f}) &= (\tilde{w} \circ \sigma)(\tilde{f} \circ \sigma), \\ \tilde{w}\tilde{f} = S^{-1}(w)T^{-1}(f) &= (w \circ \sigma^{-1})(f \circ \sigma^{-1}). \end{aligned} \tag{10}$$

For the next lemmas, w is a weight in $[0,1)$ and $\tilde{w} = S^{-1}(w)$ is the according weight in $[0,\infty)$.

Lemma 2.1. *The operators T and its inverse T^{-1} are linear positive operators and the next equalities are true:*

$$\begin{aligned} T(\tilde{\varphi}\tilde{D}^2\tilde{f}) &= \varphi D^2(T\tilde{f}), \\ T^{-1}(\varphi D^2 f) &= \tilde{\varphi}\tilde{D}^2(T^{-1}f). \end{aligned} \tag{11}$$

Proof. We will prove only the first equality (the proof of the second one is similar).
For the right side of the first equality we have

$$\begin{aligned} D(T\tilde{f}) &= D\left(\lambda(\tilde{f}\circ\sigma)\right) = -\tilde{f}\circ\sigma + \lambda D\tilde{f}\circ\sigma \\ &= -\tilde{f}\circ\sigma + \lambda\tilde{D}\tilde{f}\circ\sigma.\lambda^{-2} = -\tilde{f}\circ\sigma + \lambda^{-1}\tilde{D}\tilde{f}\circ\sigma \end{aligned}$$

and

$$\begin{aligned} D^2(T\tilde{f}) &= D\left(-\tilde{f}\circ\sigma + \lambda^{-1}\tilde{D}\tilde{f}\circ\sigma\right) \\ &= -\tilde{D}\tilde{f}\circ\sigma.\lambda^{-2} + D(\lambda^{-1})\tilde{D}\tilde{f}\circ\sigma + \lambda^{-1}D\left(\tilde{D}\tilde{f}\circ\sigma\right) \\ &= -\lambda^{-2}\tilde{D}\tilde{f}\circ\sigma + \lambda^{-2}\tilde{D}\tilde{f}\circ\sigma + \lambda^{-1}\tilde{D}^2\tilde{f}\circ\sigma.\lambda^{-2} = \lambda^{-3}\tilde{D}^2\tilde{f}\circ\sigma. \end{aligned}$$

Consequently

$$\varphi D^2(T\tilde{f}) = \lambda\frac{\varphi}{\lambda^4}\tilde{D}^2\tilde{f}\circ\sigma = \lambda\tilde{\varphi}\tilde{D}^2\tilde{f}\circ\sigma = T(\tilde{\varphi}\tilde{D}^2\tilde{f}).$$

□

Lemma 2.2. *The operator $T : C(\tilde{w})[0,\infty) \to C(w)[0,1)$ is an one-to-one correspondence with*

$$\|wT(\tilde{f})\|_{[0,1)} = \|\tilde{w}\tilde{f}\|_{[0,\infty)}, \quad \|\tilde{w}T^{-1}(f)\|_{[0,\infty)} = \|wf\|_{[0,1)}.$$

Proof. The above equalities are easily obtainable from the definition (7) of the operator T and from the equalities (10). □

Lemma 2.3. *The operator $T : W^2(\tilde{w}\tilde{\varphi})[0,\infty) \to W^2(w\varphi)[0,1)$ is an one-to-one correspondence with*

$$\|w\varphi D^2(T(\tilde{f}))\|_{[0,1)} = \|\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{f}\|_{[0,\infty)}, \quad \|\tilde{w}\tilde{\varphi}\tilde{D}^2(T^{-1}(f))\|_{[0,\infty)} = \|w\varphi D^2 f\|_{[0,1)}.$$

Proof. From the definition (7) of tne operator T and from the equalities (10) and (11) we have

$$\begin{aligned} \tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{f} &= \tilde{w}T^{-1}\left(\varphi D^2(T\tilde{f})\right) = \tilde{w}\frac{1}{\lambda\circ\sigma^{-1}}\left(\varphi D^2(T\tilde{f})\right)\circ\sigma^{-1} \\ &= \left(\lambda\circ\sigma^{-1}\right)\left(w\circ\sigma^{-1}\right)\frac{1}{\lambda\circ\sigma^{-1}}\left(\varphi D^2(T\tilde{f})\right)\circ\sigma^{-1} \\ &= \left(w\circ\sigma^{-1}\right)\left(\varphi D^2(T\tilde{f})\right)\circ\sigma^{-1} = \left(w\varphi D^2(T(\tilde{f}))\right)\circ\sigma^{-1}. \end{aligned}$$

Consequently

$$\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{f}(\tilde{x}) = \left(w\varphi D^2(T(\tilde{f}))\right)\circ\sigma^{-1}(\tilde{x}) = w\varphi D^2(T(\tilde{f}))(x)$$

or

$$\|w\varphi D^2(T(\tilde{f}))\|_{[0,1)} = \|\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{f}\|_{[0,\infty)}.$$

The proof of the second equality is similar. □

Lemma 2.4. *For every $f \in C(w)[0,1) + W^2(w\varphi)[0,1)$, $\tilde{f} = T^{-1}f$ and $t > 0$ we have*

$$K_w(f,t)_{[0,1)} = K_{\tilde{w}}(\tilde{f},t)_{[0,\infty)}.$$

Proof. From the definition of the K-functional (5) we have

$$K_{\tilde{w}}(\tilde{f},t)_{[0,\infty)} = \inf_{\tilde{g}\in W^2(\tilde{w}\tilde{\varphi}),\ \tilde{f}-\tilde{g}\in C(\tilde{w})}\left\{\|\tilde{w}(\tilde{f}-\tilde{g})\|_{[0,\infty)} + t\|\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{g}\|_{[0,\infty)}\right\}.$$

Now, from (10)

$$\tilde{w}(\tilde{f}-\tilde{g}) = (w\circ\sigma^{-1})\left((f-g)\circ\sigma^{-1}\right)$$

and consequently

$$\|\tilde{w}(\tilde{f}-\tilde{g})\|_{[0,\infty)} = \|w(f-g)\|_{[0,1)}.$$

From Lemma 2.4 we have

$$\|\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{g}\|_{[0,\infty)} = \|w\varphi D^2(T(\tilde{g}))\|_{[0,1)} = \|w\varphi D^2 g\|_{[0,1)}.$$

□

The classical Baskakov operator $V_n f(x)$ (see [1]) is defined for bounded functions $f(x)$ in $[0,\infty)$ by the formula

$$V_n f(x) = (V_n f, x) = V_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) v_{n,k}(x) \tag{12}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

The next two lemmas give the connection between the MKZ operators M_n and the Baskakov operators V_n.

Lemma 2.5. *For every f such that one of the series below is convergent and for every $n \in \mathbb{N}$ we have*

$$M_n(f)(x) = T(V_n(T^{-1}(f)))(x), \quad x \in [0,1). \tag{13}$$

Proof. From the definition of T we get

$$\begin{aligned}
&T\left(V_n\left(T^{-1}(f)\right)\right)(x) = \lambda(x)(V_n(T^{-1}(f))\circ\sigma^{-1})(x)\\
&= \frac{1}{1+\tilde{x}}(V_n(T^{-1}(f)(\tilde{x}) = \frac{1}{1+\tilde{x}}V_n(\tilde{f},\tilde{x})\\
&= \frac{1}{1+\tilde{x}}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\tilde{x}^k}{(1+\tilde{x})^{n+k}}\tilde{f}\left(\frac{k}{n}\right)\\
&= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\tilde{x}^k}{(1+\tilde{x})^{n+k+1}}\frac{1}{(\lambda\circ\sigma^{-1})\left(\frac{k}{n}\right)}(f\circ\sigma^{-1})\left(\frac{k}{n}\right).
\end{aligned}$$

Since

$$\sigma^{-1}\left(\frac{k}{n}\right) = \frac{k/n}{1+k/n} = \frac{k}{n+k}$$

we have

$$(\lambda\circ\sigma^{-1})\left(\frac{k}{n}\right) = \lambda\left(\frac{k}{n+k}\right) = \frac{n}{n+k}$$

and

$$(f\circ\sigma^{-1})\left(\frac{k}{n}\right) = f\left(\frac{k}{n+k}\right).$$

Also

$$\frac{\tilde{x}^k}{(1+\tilde{x})^{n+k+1}} = \left(\frac{\tilde{x}}{1+\tilde{x}}\right)^k\frac{1}{(1+\tilde{x})^{n+1}} = x^k(1-x)^{n+1}.$$

Consequently

$$\begin{aligned}
T(V_n(T^{-1}(f)))(x) &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{n+k}{k}x^k(1-x)^{n+1}f\left(\frac{k}{n+k}\right)\\
&= \sum_{k=0}^{\infty}\binom{n+k}{k}x^k(1-x)^{n+1}f\left(\frac{k}{n+k}\right) = M_n(f,x).
\end{aligned}$$

□

Lemma 2.6. *For every $f\in C(w)[0,1)$ and for every $n\in\mathbb{N}$ we have*

$$\left\|w(M_nf-f)\right\|_{[0,1)} = \left\|\tilde{w}(V_n\tilde{f}-\tilde{f})\right\|_{[0,\infty)}.$$

Proof. From Lemma 2.5 we have

$$\begin{aligned}
M_n(f)(x) &= T(V_n(T^{-1}(f)))(x) = \lambda(x)(V_n(T^{-1}(f))\circ\sigma^{-1})(x)\\
&= \frac{1}{1+\tilde{x}}(V_n(T^{-1}(f)(\tilde{x}) = \frac{1}{1+\tilde{x}}V_n(\tilde{f},\tilde{x}).
\end{aligned}$$

Since,

$$f(x) = T(\tilde{f})(x) = \lambda(x)(\tilde{f} \circ \sigma)(x) = \frac{1}{1+\tilde{x}}\tilde{f}(\tilde{x})$$

it follows that

$$M_n(f)(x) - f(x) = \frac{1}{1+\tilde{x}}\left(V_n(\tilde{f}, \tilde{x}) - \tilde{f}(\tilde{x})\right).$$

Also, from (8) we have

$$w(x) = S(\tilde{w})(x) = \frac{1}{\lambda(x)}(\tilde{w} \circ \sigma)(x) = (1+\tilde{x})\tilde{w}(\tilde{x}).$$

Consequently

$$\begin{aligned} w(x)\big(M_n f - f\big)(x) &= (1+\tilde{x})\tilde{w}(\tilde{x})\frac{1}{1+\tilde{x}}\left(V_n(\tilde{f}, \tilde{x}) - \tilde{f}(\tilde{x})\right) \\ &= \tilde{w}(\tilde{x})\left(V_n\tilde{f} - \tilde{f}\right)(\tilde{x}) \end{aligned}$$

i.e.

$$\big\|w(M_n f - f)\big\|_{[0,1)} = \left\|\tilde{w}(V_n\tilde{f} - \tilde{f})\right\|_{[0,\infty)}.$$

$\square$

3 Proof of Theorem 1.1 and some other results for MKZ

From Lemma 2.5 we have

$$\begin{aligned} M_n(f)(x) &= T(V_n(T^{-1}(f)))(x) = \lambda(x)(V_n(T^{-1}(f)) \circ \sigma^{-1})(x) \\ &= \frac{1}{1+\tilde{x}}(V_n(T^{-1}(f)(\tilde{x}) = \frac{1}{1+\tilde{x}}V_n(\tilde{f}, \tilde{x}). \end{aligned}$$

Since

$$f(x) = T(\tilde{f})(x) = \lambda(x)(\tilde{f} \circ \sigma)(x) = \frac{1}{1+\tilde{x}}\tilde{f}(\tilde{x})$$

it follows that

$$M_n(f)(x) - f(x) = \frac{1}{1+\tilde{x}}\left(V_n(\tilde{f}, \tilde{x}) - \tilde{f}(\tilde{x})\right).$$

Also, from (8) we have

$$w(x) = S(\tilde{w})(x) = \frac{1}{\lambda(x)}(\tilde{w} \circ \sigma)(x) = (1+\tilde{x})\tilde{w}(\tilde{x}).$$

Consequently

$$w(x)\big(M_n f - f\big)(x) = (1+\tilde{x})\tilde{w}(\tilde{x})\frac{1}{1+\tilde{x}}\left(V_n(\tilde{f},\tilde{x}) - \tilde{f}(\tilde{x})\right)$$
$$= \tilde{w}(\tilde{x})\left(V_n\tilde{f} - \tilde{f}\right)(\tilde{x})$$

i.e.

$$\big\|w(M_n f - f)\big\|_{[0,1)} = \left\|\tilde{w}(V_n\tilde{f} - \tilde{f})\right\|_{[0,\infty)}.$$

From [6][Theorem 1] we have that for weights $\tilde{w}(\tilde{x}) = \tilde{x}^{\beta_0}(1+\tilde{x})^{\beta_\infty}$, where $\beta_0 \in [-1,0], \beta_\infty \in \mathbb{R}$, the next equivalency is true, i.e.:

There exists an absolute constant L such that, for every natural number $n > L$

$$\left\|\tilde{w}(V_n\tilde{f} - \tilde{f})\right\|_{[0,\infty)} \sim K_{\tilde{w}}\left(\tilde{f}, \frac{1}{n}\right)_{[0,\infty)}.$$

From Lemma 2.4 we have

$$K_w(f,t)_{[0,1)} = K_{\tilde{w}}(\tilde{f},t)_{[0,\infty)}$$

and consequently

$$\big\|w(M_n f - f)\big\|_{[0,1)} \sim K_w\left(f, \frac{1}{n}\right)_{[0,1)}.$$

For the weights $\tilde{w}(\tilde{x}) = \tilde{x}^{\beta_0}(1+\tilde{x})^{\beta_\infty}$ we have

$$w(x) = \frac{1}{\lambda(x)}(\tilde{w}\circ\sigma)(x) = (1+\tilde{x})\tilde{w}(\tilde{x}) = \tilde{x}^{\gamma_0}(1+\tilde{x})^{\gamma_\infty+1}$$
$$= x^{\gamma_0}(1-x)^{-(\gamma_\infty+\gamma_0+1)} = x^{\gamma_0}(1-x)^{\gamma_1}.$$

Since $\beta_0 \in [-1,0], \beta_\infty \in \mathbb{R}$ we have $\gamma_0 \in [-1,0], \gamma_1 \in \mathbb{R}$.

The proof of Theorem 1.1 is complete.

From Lemma 2.6, Lemma 2.3 and Lemma 5 in [6] we obtain the next Jackson-type inequality.

Theorem 3.1. *For w, defined by* (2) *there exists a constant C such that for every natural $n \geq |1+\gamma_0+\gamma_1|$ we have*

$$\big\|w(M_n f - f)\big\|_{[0,1)} \leq \frac{C}{n}\left\|w\varphi D^2 f\right\|_{[0,1)}$$

for every function $f \in W^2(w\varphi)[0,1)$.

From the definition of T, Lemma 2.3, Lemma 2.5 and Lemma 7 in [6] we obtain the next Bernstein-type inequality.

Theorem 3.2. *For w, defined by* (2) *there exists a constant C such that for every natural $n \geq |1+\gamma_0+\gamma_1|$ we have*

$$\left\|w\varphi D^2 M_n f\right\|_{[0,1)} \leq Cn\|wf\|_{[0,1)}$$

for every function $f \in C(w)[0,1)$.

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