Weighted Approximation of functions in $L_\infty[0,\infty)$ [1]

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Abstract

Direct theorems in terms of the weighted K-functional for the uniform weighted approximation by a class of operators which reproduce the functions $E_i(x)=\dfrac{x^i}{1+x}, i=0,1$ are obtained for functions from $C(w)+W^2_\mu(w\phi)$ with weights of the form $\left(\frac{x}{1+x}\right)^{\beta_0}\left(\frac{1}{1+x}\right)^{\beta_\infty}$ for $\beta_0,\beta_\infty\in[-1,0]$. As a consequence, direct theorems for some (for instance, classical and Goodman-Sharma modifications of Baskakov and Meyer-König and Zeller) operators are obtained.

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1. Introduction

In order to approximate functions in $[0,\infty)$, Baskakov (in analogy with the Bernstein operator) introduced a new operator (see [1]). It is defined for bounded functions $f(x)$ in $[0,\infty)$ by the formula

$$B_nf(x)=\sum_{k=0}^{\infty}P_{n,k}(x)f\left(\frac{k}{n}\right) \tag{1.1}$$

where

$$P_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k}.$$

One way to generalize this is to replace $f\left(\frac{k}{n}\right)$ in the above definition by some functionals $b_{n,k}(f)$, defined for every $f\in L_\infty[0,\infty)$ and satisfying given conditions. A different path is to consider a sequence of linear positive operators and impose appropriate conditions.

In this paper we investigate the approximation of functions $f\in L_\infty[0,\infty)$ by a sequence of operators L_n which satisfy the next conditions:

$$L_n \quad \text{are linear and positive operators,} \tag{1.2}$$
$$L_n(E_i,x)=E_i(x) \text{ for i=0 and i=1,} \tag{1.3}$$
$$L_n(E_2,x)=E_2(x)+A_n(x), \tag{1.4}$$
$$L_n(E_0^2,x)=E_0^2(x)+B_n(x). \tag{1.5}$$

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Here by $L_\infty[0,\infty)$ we denote the space of all Lebesgue measurable and essentially bounded in $[0,\infty)$ functions equipped with the uniform norm $\|\cdot\|$, E_i (for $i = 0, 1, 2$) are the functions $E_i(x) = \dfrac{x^i}{1+x}$ and the functions $A_n(x)$ and $B_n(x)$ are such that for every $x \in [0,\infty)$ we have $|A_n(x)| \le a_n x$, $|B_n(x)| \le b_n E_0^2(x)$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

And we note here that if the above conditions are satisfied, then automatically we have also

$$L_n(1,x) = 1 \tag{1.6}$$
$$L_n(E_1^2,x) = E_1^2(x) + B_n(x). \tag{1.7}$$

We define an appropriate K-functional and by using it we prove a direct theorem for them. But before formulating the main result we will start with some definitions and notations.

The first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^2g(x) = g''(x)$. By $\phi(x) = x$ we denote the weight which is naturally connected with the second derivatives of these operators. Our main goal in this paper is the characterization of L_∞-norm of the weighted approximation error $\|w(f - L_n f)\|$ for weight functions given by

$$w(x) = w_\beta(x) = w(\beta_0,\beta_\infty;x) = \left(\frac{x}{1+x}\right)^{\beta_0}\left(\frac{1}{1+x}\right)^{\beta_\infty}. \tag{1.8}$$

where $x \in [0,\infty)$ and $\beta_0, \beta_\infty \in [-1,0]$.

By $C[0,\infty)$ we denote the space of all continuous on $[0,\infty)$ functions. The functions from $C[0,\infty)$ are not expected to be bounded or uniformly continuous.

For a weight function w we set

$$C(w)[0,\infty) = C(w) = \{f \in C[0,\infty) : wf \in L_\infty[0,\infty)\}$$

and

$$W_\mu^2(w\phi)[0,\infty) = W_\mu^2(w\phi) = \left\{ g \,:\, \mu g, D(\mu g) \in AC_{loc}(0,\infty), w\phi D^2(\mu g) \in L_\infty[0,\infty)\right\},$$

where $AC_{loc}(0,\infty)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,\infty)$ and $\mu(x) = 1+x$.

The weighted approximation error of L_n will be compared with the K-functional between the weighted spaces $C(w)$ and $W_\mu^2(w\phi)$, which for every

$$f \in C(w) + W_\mu^2(w\phi) = \{f_1 + f_2 : f_1 \in C(w), f_2 \in W_\mu^2(w\phi)\}$$

and $t > 0$ is defined by

$$K_w(f,t) = \inf\left\{\|w(f-g)\| + t\|w\phi D^2(\mu g)\| : g \in W_\mu^2(w\phi), f-g \in C(w)\right\}. \tag{1.9}$$

The above formula is a standard definition of K-functional in interpolation theory. In approximation theory the condition $f - g \in C(w)$ in (1.9) is usually omitted because in the predominant number of cases the second interpolation space is embedded in the first one. However, in this case we have interpolation between $C(w)$ and $W_\mu^2(w\phi)$, as $W_\mu^2(w\phi)\backslash C(w)$ is of infinite dimension for some of the weights w that satisfy the above assumptions.

Our main result is a direct inequality. It is a modification of the result in [10], which treats the case of a class of Bernstein-type operators.

Theorem 1.1. *Let the operators* L_n *satisfy the conditions* (1.2) - (1.5) *and* w *is given by* (1.8) *with* $\beta_0, \beta_\infty \in [-1,0]$ *and* $c_n = \sqrt{2} max\{a_n, \sqrt{a_n b_n}\}$. *Then for every* $f \in C(w) + W_\mu^2(w\phi)$ *and for every* $n \in \mathbb{N}$ *we have*

$$\|w(L_n f - f)\| \leq 2K_w\left(f, c_n\right).$$

Some remarks follow.

Remark 1.

Let $\Pi_1 = \{f : f = aE_0 + bE_1, a, b \in \mathbb{R}\}$. Then for $\beta_0 \in [-1,0], \beta_\infty \in (-1,0)$ the space $C(w) + W_\mu^2(w\phi)$ coincides with the space $C(w) + \Pi_1$. Indeed, let $f \in C(w) + W_\mu^2(w\phi)$, i.e. f can be written as $f = f_1 + f_2$ where $f_1 \in C(w)$ and $f_2 \in W_\mu^2(w\phi)$. Then we have $(\mu f_2)(x) = a^* x + b^* + g^*(x)$ where

$$g^*(x) = -\int_0^x \int_v^\infty (\mu f_2)''(u)dudv$$

and

$$b^* = (\mu f_2)(0), \quad a^* = (\mu f_2)'(\infty) := (\mu f_2)'(1) + \int_1^\infty (\mu f_2)''(v)dv.$$

Obviously, $g^* \in C(w\phi)$ and $g = \mu^{-1} g^* \in C(w)$ and the above follows from

$$f_2(x) = \frac{a^* x + b^*}{\mu} + \frac{g^*(x)}{\mu} = aE_0(x) + bE_1(x) + g(x).$$

But for $\beta_\infty = 0$ or for $\beta_\infty = -1$ the space $C(w) + W_\mu^2(w\phi)$ is essentially bigger than $C(w) + \Pi_1$. For instance, for the function $f(x) = \log(1+x)$, we have $f \in (C(w) + W_\mu^2(w\phi)) \backslash (C(w) + \Pi_1)$ for $\beta_\infty = 0$ and $\beta_0 \in [-1,0]$. The same is true for the function $f(x) = (1+x)\log(1+x)$ for $\beta_\infty = -1$ and $\beta_0 \in [-1,0]$.

Remark 2.

Theorem (1.1) does not imply for all $f \in C(w) + W_\mu^2(w\phi)$ that $\|w(L_n f - f)\| \to 0$ or $K_w\left(f, c_n\right) \to 0$ when $n \to \infty$. Actually, none of these quantities tends to zero with $n \to \infty$ for some functions $f \in C(w)$. In order to ensure convergence to zero of these quantities one may need to impose additional restrictions on the behavior of f at 0 and at ∞. At 0 these restrictions are (see [4]) $\lim_{x\to 0+} x^{\beta_0} f(x) = 0$ for $-1 < \beta_0 < 0$ or the existence of $\lim_{x\to 0+} x^{-1} f(x)$ for $\beta_0 = -1$. In the same time, at ∞ the restrictions are more complicated but, shortly, the function f should not vary very fast in order to allow approximation in $C(w)$ with functions from $W_\mu^2(w\phi)$.

2. Main result

It is well known fact that if an operator $\tilde{L}$ satisfies the following two conditions:

$$\tilde{L} \quad \text{is linear and positive operator;} \tag{2.1}$$

$$\tilde{L}(1,x) = 1 \ , \ \ \tilde{L}(t,x) = x; \tag{2.2}$$

then for every concave continuous function f the next inequality is true

$$f \geq \tilde{L} f. \tag{2.3}$$

As a consequence of this we have that if an operator L satisfies the following two conditions:

$$L \quad \text{is linear and positive operator;} \tag{2.4}$$
$$L(E_0, x) = E_0(x) \ \ , \ \ L(E_1, x) = E_1(x); \tag{2.5}$$

then the operator $\mu L\left(\frac{f}{\mu}\right)$ satisfies the conditions (2.1) and (2.2) and consequently for every concave continuous function f the next inequality is true

$$\frac{f}{\mu} \geq L\left(\frac{f}{\mu}\right). \tag{2.6}$$

Now we prove two lemmas which we will need later.

Lemma 2.1. For every function $f \in C(w)[0, \infty)$ we have $\|wL(f)\| \leq \| wf\|$, i.e. the norm of the operator is 1.

Proof. Let us mention that the function $\mu(w)^{-1}$ is concave for $\beta_0, \beta_\infty \in [-1, 0]$ and then from (2.6) we have

$$L\left((w)^{-1}\right) = L\left(\frac{\mu(w)^{-1}}{\mu}\right) \leq (w)^{-1}.$$

This one, (2.4) and (2.5) give

$$\begin{aligned} \|wL(f)\| &= \|wL\left(wf(w)^{-1}\right)\| \\ &\leq \|wf\| \, \|wL\left((w)^{-1}\right)\| \\ &\leq \|wf\| \, \|w(w)^{-1}\| = \|wf\|. \end{aligned}$$

□

Lemma 2.2. For $\beta_0, \beta_\infty \in [-1, 0]$ and $x, t \in (0, \infty)$

$$\left|\int_x^t \frac{t-u}{\phi(u)w(u)} du\right| \leq 2\sqrt{2} \max\left\{\frac{(x-t)^2}{\phi(x)w(x)}, \frac{(x-t)^2}{\sqrt{1+t\sqrt{x}}w(x)}\right\}.$$

Proof. Case 1. $t \geq x$.

It is obvious that $\dfrac{1}{\phi(u)w(u)} \leq \dfrac{1}{\phi(x)w(x)}$ for $u \in [x, t]$ because the function $\phi(u)w(u) = u^{1+\beta_0}(1+u)^{-\beta_0-\beta_\infty}$ is monotonically increasing.

Then we have

$$\left|\int_x^t \frac{t-u}{\phi(u)w(u)} du\right| \leq \frac{(t-x)^2}{2\phi(x)w(x)}. \tag{2.7}$$

Case 2.1. $t < x$ and $\beta_\infty \in \left[-\frac{1}{2}, 0\right]$.

In this case the powers $1+\beta_0$ and $-\beta_0-\beta_\infty$ are positive and consequently

$$\begin{aligned}
&\left|\int_x^t \frac{t-u}{\phi(u)w(u)}du\right| = \int_t^x \frac{u-t}{\phi(u)w(u)}du \\
&= \int_t^x \frac{u-t}{u^{1+\beta_0}(1+u)^{-\beta_0-\beta_\infty}}du \\
&= \int_t^x (u-t)^{1-(1+\beta_0)-(-\beta_0-\beta_\infty)} \left(\frac{u-t}{u}\right)^{1+\beta_0} \left(\frac{u-t}{u+1}\right)^{-\beta_0-\beta_\infty} du \\
&= \int_t^x (u-t)^{\beta_\infty} \left(1-\frac{t}{u}\right)^{1+\beta_0} \left(1-\frac{t+1}{u+1}\right)^{-\beta_0-\beta_\infty} du \\
&\le \int_t^x (u-t)^{\beta_\infty} \left(1-\frac{t}{x}\right)^{1+\beta_0} \left(1-\frac{t+1}{x+1}\right)^{-\beta_0-\beta_\infty} du \\
&= \left(1-\frac{t}{x}\right)^{1+\beta_0} \left(1-\frac{t+1}{x+1}\right)^{-\beta_0-\beta_\infty} \int_t^x (u-t)^{\beta_\infty} du \\
&= \frac{(x-t)^{1-\beta_\infty}}{x^{1+\beta_0}(1+x)^{-\beta_0-\beta_\infty}} \int_t^x (u-t)^{\beta_\infty} du \\
&= \frac{(x-t)^{1-\beta_\infty}}{\phi(x)w(x)}(1+\beta_\infty)^{-1}(x-t)^{1+\beta_\infty} \\
&= (1+\beta_\infty)^{-1}\frac{(x-t)^2}{\phi(x)w(x)} \\
&\le 2\frac{(x-t)^2}{\phi(x)w(x)}. \qquad (2.8)
\end{aligned}$$

Case 2.2. $t < x$ and $\beta_\infty \in \left[-1, -\frac{1}{2}\right]$.

In this case $\frac{1}{(1+u)^{1/2}} \le \frac{1}{(1+t)^{1/2}}$ for $u \in [t,x]$ and the powers $1+\beta_0$ and $-\beta_0-\beta_\infty-1/2$ are positive. In the same way as in *Case 2.1.* we have

$$\begin{aligned}
&\left|\int_x^t \frac{t-u}{\phi(u)w(u)}du\right| = \int_t^x \frac{u-t}{\phi(u)w(u)}du \\
&= \int_t^x \frac{u-t}{u^{1+\beta_0}(1+u)^{-\beta_0-\beta_\infty}}du \\
&\le (1+t)^{-1/2}\int_t^x \frac{u-t}{u^{1+\beta_0}(1+u)^{-\beta_0-\beta_\infty-1/2}}du \\
&= (1+t)^{-1/2}\int_t^x (u-t)^{\beta_\infty+1/2} \left(1-\frac{t}{u}\right)^{1+\beta_0} \left(1-\frac{t+1}{u+1}\right)^{-\beta_0-\beta_\infty-1/2} du \\
&\le (1+t)^{-1/2}\int_t^x (u-t)^{\beta_\infty+1/2} \left(1-\frac{t}{x}\right)^{1+\beta_0} \left(1-\frac{t+1}{x+1}\right)^{-\beta_0-\beta_\infty-1/2} du \\
&= (1+t)^{-1/2}\left(1-\frac{t}{x}\right)^{1+\beta_0} \left(1-\frac{t+1}{x+1}\right)^{-\beta_0-\beta_\infty-1/2} \int_t^x (u-t)^{\beta_\infty+1/2} du \\
&= \frac{(x-t)^{1/2-\beta_\infty}}{\sqrt{1+t}\,x^{1+\beta_0}(1+x)^{-\beta_0-\beta_\infty-1/2}} \int_t^x (u-t)^{\beta_\infty+1/2} du \\
&= \sqrt{\frac{1+x}{1+t}}\frac{(x-t)^{1/2-\beta_\infty}}{\phi(x)w(x)}\left(\frac{3}{2}+\beta_\infty\right)^{-1}(x-t)^{3/2+\beta_\infty}
\end{aligned}$$

$$= \left(\frac{3}{2} + \beta_\infty\right)^{-1} \sqrt{\frac{1+x}{1+t}} \frac{(x-t)^2}{\phi(x)w(x)}$$
$$\leq 2\sqrt{\frac{1+x}{1+t}} \frac{(x-t)^2}{\phi(x)w(x)}.$$

Here we consider two subcases for x. For $x \in (0, 1]$ we have $\sqrt{\frac{1+x}{1+t}} \leq \sqrt{2}$ and from the above we obtain

$$\left| \int_x^t \frac{t-u}{\phi(u)w(u)} du \right| \leq 2\sqrt{2} \frac{(x-t)^2}{\phi(x)w(x)}. \tag{2.9}$$

For $x \in [1, \infty)$ we have $\frac{\sqrt{1+x}}{x} \leq \sqrt{\frac{2}{x}}$ and consequently

$$\left| \int_x^t \frac{t-u}{\phi(u)w(u)} du \right| \leq 2\sqrt{2} \frac{(x-t)^2}{\sqrt{1+t}\sqrt{x}w(x)}. \tag{2.10}$$

From (2.9) and (2.10) we have

$$\left| \int_x^t \frac{t-u}{\phi(u)w(u)} du \right| \leq \max \left\{ 2\sqrt{2} \frac{(x-t)^2}{\phi(x)w(x)},\ 2\sqrt{2} \frac{(x-t)^2}{\sqrt{1+t}\sqrt{x}w(x)} \right\}. \tag{2.11}$$

Finaly from (2.7), (2.8) and (2.11) we get

$$\left| \int_x^t \frac{t-u}{\phi(u)w(u)} du \right| \leq 2\sqrt{2} \max \left\{ \frac{(x-t)^2}{\phi(x)w(x)},\ \frac{(x-t)^2}{\sqrt{1+t}\sqrt{x}w(x)} \right\}.$$

□

The result from the above lemma we use in the next theorem.

Theorem 2.1. *(Jackson-type inequality). Let us define for an operator L the quantities*

$$\mathcal{A} = \|\phi^{-1}\left(L(E_2) - E_2\right)\| \quad \textit{and} \quad \mathcal{B} = \|\mu^2\left(L(E_1^2) - E_1^2\right)\|.$$

Then for every operator L which satisfies conditions 2.4 and 2.5 and such that $\mathcal{A}$ is finite and for every function $f \in W_\mu^2(w\phi)$ we have

$$\|w(Lf - f)\| \leq 2\sqrt{2}\left\|w\phi D^2(\mu f)\right\| \max\left\{\mathcal{A},\ \sqrt{\mathcal{A}.\mathcal{B}}\right\}.$$

Proof. Let $g(z) = \mu(z)f(z)$. Then by using Taylor's formula

$$g(t) = g(x) + (t-x)Dg(x) + \int_x^t (t-v)D^2g(v)dv.$$

we have for $f(x)$

$$f(t) = f(x) + \frac{t-x}{1+t}(1+x)Df(x) + \frac{1}{1+t}\int_x^t (t-v)D^2\left(\mu(v)f(v)\right)dv.$$

Applying operator L to both sides of the above equality and using (2.4), (2.5) and Lemma 2.2 we obtain

$$|L(f,x)-f(x)| = \left| L\left(\frac{1}{1+t}\int_x^t (t-v)D^2\left(\mu(v)f(v)\right)dv,\ x\right)\right| \tag{2.12}$$
$$\leq L\left(\frac{1}{1+t}\left|\int_x^t \frac{t-v}{\phi(v)w(v)}dv\right|,\ x\right)\|w\phi D^2(\mu f)\|$$
$$\leq \left\|w\phi D^2(\mu f)\right\| L\left(\max\left\{2\sqrt{2}\frac{(x-t)^2}{(1+t)\phi(x)w(x)},\ 2\sqrt{2}\frac{(x-t)^2}{(1+t)^{3/2}\sqrt{x}w(x)}\right\},\ x\right)$$
$$\leq 2\sqrt{2}\frac{\|w\phi D^2(\mu f)\|}{w(x)}\max\left\{L\left(\frac{(x-t)^2}{(1+t)\phi(x)},\ x\right),\ L\left(\frac{(x-t)^2}{\sqrt{x}(1+t)^{3/2}},\ x\right)\right\}.$$

Applying (2.5) for the first term and Cauchy's inequality and (2.5) for the second term in right hand side of (2.12) we have

$$L\left(\frac{(x-t)^2}{(1+t)\phi(x)},\ x\right) = \phi^{-1}(x)\left(L(E_2,x)-E_2(x)\right)$$

and

$$L\left(\frac{(x-t)^2}{\sqrt{x}(1+t)^{3/2}},\ x\right) \leq \left(L\left(\frac{(x-t)^2}{(1+t)\phi(x)},\ x\right)\right)^{1/2}.\left(L\left(\frac{(x-t)^2}{(1+t)^2},\ x\right)\right)^{1/2}$$
$$= \left(\phi^{-1}(x)\left(L(E_2,x)-E_2(x)\right)\right)^{1/2}.\left(\mu^2(x)\left(L(E_1^2,x)-E_1^2(x)\right)\right)^{1/2}.$$

Replacing the above two estimations in (2.12) we have

$$w(x)\left|L(f,x)-g(x)\right| \leq 2\sqrt{2}\left\|w\phi D^2(\mu f)\right\|.\max\left\{\mathcal{A}(x),\ \sqrt{\mathcal{A}(x).\mathcal{B}(x)}\right\}. \tag{2.13}$$

Taking a supremum on x in (2.13) we complete the proof of Theorem 2.1.

□

As an elementary consequence of this lemma we have that if a function $f \in W_\mu^2(w\phi)$ then $Lf - f \in C(w)$.

From 1.4 and 1.5 it follows that

$$\left|\phi^{-1}(x)\left(L_n(E_2,x)-E_2(x)\right)\right| \leq a_n,$$
$$\left|\mu^2(x)\left(L_n(E_1^2,x)-E_1^2(x)\right)\right| \leq b_n.$$

Above result and Theorem 2.4 give

Theorem 2.2. *For every function $f \in W_\mu^2(w\phi)$ we have*

$$\|w(L_nf-f)\| \leq 2c_n\left\|w\phi D^2(\mu f)\right\|.$$

We use Theorem 2.2 in the proof of Theorem 1.1.

Proof. (*Theorem 1.1*)
Let g is an arbitrary function in $W^2_\mu(w\phi)$, such that $g - f \in C(w)$. Then

$$\|w(L_n f - f)\| \le \|w(L_n f - L_n g)\| + \|w(L_n g - g)\| + \|w(g - f)\|.$$

From Lemma 2.1 and Theorem 2.5 we get

$$\begin{aligned}\|w(L_n f - f)\| &\le 2\|w(f - g)\| + 2c_n \left\|w\phi D^2(\mu f)\right\| \\ &= 2\left(\|w(f - g)\| + c_n \left\|w\phi D^2(\mu f)\right\|\right).\end{aligned}$$

Taking infimum on all $g \in W^2_\mu(w\phi)$ such that $(f - g) \in C(w)$ in the above inequality we complete the proof of Theorem 1.1. □

Now we can easily prove a similar result for linear positive operators which reproduce linear functions.

Let us denote the basic test functions by e_i, i.e. $e_i(x) = x^i$ for $i = 0, 1, 2$ and the weight by $\psi(x) = x(1 + x)$. Let the sequence of linear positive operators L_n satisfy the next conditions

$$L_n(e_i, x) = e_i(x), \quad i = 0, 1, \tag{2.14}$$

$$L_n(e_2, x) = e_2(x) + Q_n(x), \tag{2.15}$$

$$L_n(E_0, x) = E_0(x) + R_n(x), \tag{2.16}$$

where $|Q_n(x)| \le q_n\psi(x)$, $|R_n(x)| \le r_n E_0(x)$ and $\lim\limits_{n\to\infty} q_n = \lim\limits_{n\to\infty} r_n = 0$.

The weights in consideration are

$$w_\gamma(x) = w_\gamma(\gamma_0, \gamma_\infty; x) = \left(\frac{x}{1 + x}\right)^{\gamma_0} (1 + x)^{\gamma_\infty}, \quad \gamma_0, \gamma_\infty \in \mathbb{R}. \tag{2.17}$$

We also set

$$W^2(w_\gamma\psi) = \left\{g, g' \in AC_{loc}(0, \infty) : w_\gamma\psi D^2 g \in L_\infty[0, \infty)\right\}.$$

and for $t > 0$ and every function $f \in C(w_\gamma) + W^2(w_\gamma\psi)$ a K-functional $K_{w_\gamma}(f, t)$ by

$$K_{w_\gamma}(f, t) = \inf_{g\in W^2(w_\gamma\psi),\ f-g\in C(w_\gamma)} \left\{\|w_\gamma(f - g)\|_{[0,\infty)} + t\|w_\gamma\psi D^2 g\|_{[0,\infty)}\right\}.$$

It is not difficult to see that the operators $\frac{1}{\mu(x)}L_n(\mu f, x)$ satisfy the conditions (1.3 - 1.5) with $q_n = a_n$ and $r_n = b_n$. Applying Theorem (1.1) for the operators $\frac{1}{\mu(x)}L_n(\mu f, x)$, the function $\frac{f}{\mu}$ and the weight μw_γ we obtain

Theorem 2.3. *Let w_γ be given by (2.17) with $\gamma_0, \gamma_\infty \in [-1, 0]$ and $d_n = \sqrt{2} max\{q_n, \sqrt{q_n r_n}\}$. Then for every $f \in C(w_\gamma) + W^2(w_\gamma\psi)$ and every $n \in \mathbb{N}$ we have*

$$\|w_\gamma(L_n f - f)\| \le 2K_{w_\gamma}(f, d_n).$$

3. Applications

3.1. **Immediate applications**

–A slight modification of classical Baskakov operator

For bounded and continuous on $[0, \infty)$ functions f and natural $n \geq 2$ we consider linear positive operator

$$B_n^{[sl]}(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x) f\left(\frac{k}{n-1}\right).$$

It is easy to see that the conditions 1.3, 1.4 and 1.5 are satisfied with $a_n = \frac{1}{n-1}$, $b_n \leq \frac{1}{n-2}$ and the result of Theorem 1.1 is

Proposition 3.1. *For every $f \in C(w)[0,\infty) + W_\mu^2(w\phi)[0,\infty)$ and every $n \in \mathbb{N}$, $n \geq 3$, we have*

$$\|w(f - B_n^{[sl]} f)\| \leq 2K_w\left(f, \sqrt{\frac{2}{(n-1)(n-2)}}\right) \leq 2K_w\left(f, \frac{\sqrt{2}}{n-2}\right).$$

– Agrawal and Thamer operators

The Durrmeyer-Baskakov-type operator is given for every natural n by

$$\begin{aligned} B_n^{[AT]}(f,x) &= \sum_{k=0}^{\infty} P_{n,k}(x) b_{n,k}(f), \\ b_{n,0}(f) = f(0);\quad b_{n,k}(f) &= (n-1)\int_0^\infty P_{n,k-1}(y) f(y)\, dy, \quad k \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where f is Lebesgue measurable on $(0,\infty)$ with a finite limit $f(0)$ at 0. The modification was introduced by Agrawal and Thamer [2]. Here conditions 1.3, 1.4 and 1.5 are satisfied with $a_n = \frac{2}{n-2}$, $b_n \leq \frac{2}{n-2}$ and the result of Theorem 1.1 is

Proposition 3.2. *For every $f \in C(w)[0,\infty) + W_\mu^2(w\phi)[0,\infty)$ and every $n \in \mathbb{N}$, $n \geq 3$, we have*

$$\|w(f - B_n^{[AT]} f)\| \leq 2K_w\left(f, \frac{2\sqrt{2}}{n-2}\right).$$

3.2. **Applications for Baskakov-type Operators that preserve linear functions**

–Classical Baskakov operator

It is easy to see that the operator B_n defined by (1.1) satisfy the conditions (2.14), (2.15) and (2.16) with $q_n = 1/n$ and $r_n = 1/(n-1)$. So, applying Theorem (2.3) we have

Proposition 3.3. *Let w_γ be given by (2.17) with $\gamma_0, \gamma_\infty \in [-1,0]$. Then for every $f \in C(w_\gamma)[0,\infty) + W^2(w_\gamma\psi)[0,\infty)$ and every $n \in \mathbb{N}$, $n \geq 2$, we have*

$$\|w_\gamma(f - B_n f)\| \leq 2K_{w_\gamma}\left(f, \frac{\sqrt{2}}{n-1}\right).$$

Remarks.

1.Actualy

$$\frac{1}{\mu(x)} B_n(\mu f, x) = B^{[sl]}_{n+1}(f,x), \quad x \in [0,\infty).$$

2.More general direct result and a strong converse result of type A are obtained in [6] using different arguments:

Proposition 3.4. *Let w_γ be given by* (2.17) *with* $\gamma_0 \in [-1,0]$, $\gamma_\infty \in \mathbb{R}$. *Then there exists positive constants C_1, C_2 and L such that for every*
$f \in C(w_\gamma)[0,\infty) + W^2(w_\gamma\psi)[0,\infty)$ *and every $n \in \mathbb{N}$, $n \geq L$, we have*

$$C_1 \|w_\gamma(f - B_n f)\| \leq K_{w_\gamma}\left(f, \frac{1}{n}\right) \leq C_2 \|w_\gamma(f - B_n f)\|.$$

–A Goodman-Sharma modification of classical Baskakov operator

Finta [5] introduced in 2005 the operator

$$V_n(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x) v_{n,k}(f),$$

$$v_{n,0}(f) = f(0); \quad v_{n,k}(f) = (n+1)\int_0^\infty P_{n+2,k-1}(y) f(y)\, dy, \quad k \in \mathbb{N},$$

where f is Lebesgue measurable on $(0,\infty)$ with a finite limit $f(0)$ at 0.

The operator V_n satisfy the conditions (2.14), (2.15) and (2.16) with $q_n = 2/(n-1)$ and $r_n = 2/(n-1)$. So, applying applying Theorem (2.3) we have

Proposition 3.5. *Let w_γ be given by* (2.17) *with* $\gamma_0, \gamma_\infty \in [-1,0]$. *Then for every* $f \in C(w_\gamma)[0,\infty) + W^2(w_\gamma\psi)[0,\infty)$ *and every $n \in \mathbb{N}$, $n \geq 2$, we have*

$$\|w_\gamma(f - V_n f)\| \leq 2K_{w_\gamma}\left(f, \frac{2\sqrt{2}}{n-1}\right).$$

Remarks.

1.Actualy

$$\frac{1}{\mu(x)} V_n(\mu f, x) = B^{[AT]}_{n+1}(f,x), \quad x \in [0,\infty).$$

2.A better direct result and a strong converse result of type A are obtained in [7] using different arguments:

Proposition 3.6. *Let w_γ be given by* (2.17) *with* $\gamma_0, \gamma_\infty \in [-1,0]$. *Then for every* $f \in C(w_\gamma) + W^2(w_\gamma\psi)$ *and every $n \in \mathbb{N}$, $n \geq 4$, we have*

$$\|w_\gamma(f - V_n f)\| \leq 2K_{w_\gamma}\left(f, \frac{1}{2n}\right) \leq 13.7\|w_\gamma(f - V_n f)\|.$$

–A Baskakov-Szasz-Durrmeyer operator

In [11] Gupta and Srivastava proposed the Durrmeyer-type Baskakov-Szasz operator as

$$V_n^*(f,x) = n\sum_{k=0}^{\infty} P_{n,k}(x) \int_0^\infty s_{n,k}(t) f(t) dt,$$

where

$$s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}$$

are the basic Szasz-Mirakjan polynomials. But this operator does not preserve linear functions. So, we slightly modify V_n^* in a Goodman-Sharma way and consider the operator

$$\tilde{V}_n(f,x) = f(0)(1+x)^{-n} + n\sum_{k=1}^{\infty} P_{n,k}(x)\int_0^\infty s_{n,k-1}(t)f(t)dt.$$

It is easy to see that $\tilde{V}_n$ satisfy conditions 2.14 and 2.15 with $Q_n(x) = \frac{x(x+2)}{n}$. From here we have $q_n \le \frac{2}{n}$.

Now we will prove that $r_n \le \frac{2}{n-1}$. Indeed, since the function E_0 is convex and the operator $\tilde{V}_n$ is linear and positive and reproduces linear functions it follows that

$$E_0(x) \le \tilde{V}_n\left(E_0, x\right). \tag{3.2}$$

At the same time we have for $k = 1$

$$n\int_0^\infty e^{-nt}\frac{dt}{1+t} \le n\int_0^\infty e^{-nt}dt = 1 = \frac{n+1}{n+1} \tag{3.3}$$

and for $k \ge 2$,

$$I_k = n\int_0^\infty s_{n,k-1}(t)\frac{dt}{1+t} = \frac{n}{k-1} - \frac{n}{k-1}I_{k-1} \tag{3.4}$$

and

$$I_k = I_{k-1} - \int_0^\infty s_{n,k-1}(t)\frac{dt}{(1+t)^2}. \tag{3.5}$$

Multiplying (3.5) by $\frac{n}{k-1}$ and summing with (3.4) we obtain

$$I_k = \frac{n}{n+k-1} - \frac{1}{n+k-1}\int_0^\infty s_{n,k-1}(t)\frac{dt}{(1+t)^2} \le \frac{n+1}{n+k}. \tag{3.6}$$

From (3.3) and (3.6) it follows that for every $k \in \mathbb{N}$

$$n\int_0^\infty s_{n,k-1}(t)\frac{dt}{1+t} \le \left(1+\frac{1}{n}\right)\frac{n}{n+k}. \tag{3.7}$$

Consequently,

$$\begin{aligned}\tilde{V}_n\left(\frac{1}{1+t}, x\right) &\le \left(1+\frac{1}{n}\right)\sum_{k=0}^{\infty} P_{n,k}(x)\frac{1}{1+k/n} = \left(1+\frac{1}{n}\right)B_n\left(\frac{1}{1+t}, x\right)\\ &\le \left(1+\frac{1}{n}\right)\frac{n}{n-1}\frac{1}{1+x} = \frac{n+1}{n-1}\frac{1}{1+x} = E_0 + \frac{2}{n-1}E_0.\end{aligned}$$

So, applying Theorem (2.3) for the operator $\tilde{V}_n$ we obtain

Proposition 3.7. *Let w_γ be given by (2.17) with $\gamma_0, \gamma_\infty \in [-1, 0]$. Then for every $f \in C(w_\gamma)[0,\infty) + W^2(w_\gamma\psi)[0,\infty)$ and every $n \in \mathbb{N}$, $n \ge 2$, we have*

$$\|w_\gamma(f - \tilde{V}_n f)\| \le 2K_{w_\gamma}\left(f, \frac{2\sqrt{2}}{n-1}\right).$$

3.3. **Applications for Meyer-König and Zeller-type operators that preserve linear functions**

–Classical Meyer-König and Zeller operators

The Meyer-König and Zeller operator (introduced in 1960 [9]) in the slight modification of Cheney and Sharma [3] is defined for $f \in C[0,1)$ by

$$M_n f(x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{k+n}\right), \quad x \in [0,1),$$

where $m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}$.

We shall utilize the change of variable $\sigma : [0,\infty) \to [0,1)$ given by

$$x = \sigma(\tilde{x}) = \frac{\tilde{x}}{1+\tilde{x}}.$$

A function $\tilde{f}$ defined on $[0,\infty)$ is transformed to a function f defined on $[0,1)$ by

$$f(x) = f(\sigma(\tilde{x})) = \tilde{f}(\tilde{x}).$$

The weights in consideration are

$$w_\alpha(x) = w_\alpha(\alpha_0, \alpha_1; x) = x^{\alpha_0}(1-x)^{\alpha_1}, \tag{3.8}$$

and $\varphi(x) = x(1-x)^2$ for $x \in [0,1)$ and real α_0, α_1.

We also set $W^2(w_\alpha \varphi)[0,1) = \{g, g' \in AC_{loc}(0,1) : w_\alpha \varphi D^2 g \in L_\infty[0,1)\}$.

By analogy with (1.9) we define the K-functional between the weighted spaces $C(w_\alpha)$ and $W^2(w_\alpha\varphi)$, which for every $f \in C(w_\alpha)[0,1) + W^2(w_\alpha\varphi)[0,1)$ and $t > 0$ is defined by

$$K_{w_\alpha}(f,t)_{[0,1)} = \inf_{g \in W^2(w_\alpha\varphi),\ f-g \in C(w_\alpha)} \left\{ \|w_\alpha(f-g)\|_{[0,1)} + t\|w_\alpha \varphi D^2 g\|_{[0,1)} \right\}.$$

Using the equalities

$$B_{n+1}^{[sl]}(\tilde{f}, \tilde{x}) = M_n(f, x),$$
$$\phi(\tilde{x}) D^2 \left(\mu(\tilde{x})\tilde{f}(\tilde{x})\right) = \varphi(x) D^2 f(x),$$

the relations between the weights with tildes and without tildes and the result in Proposition 3.1 we have

Proposition 3.8. *Let w_α be given by* (3.8) *with $\alpha_0, \alpha_1 \in [-1,0]$. Then for every $f \in C(w_\alpha)[0,1) + W^2(w_\alpha\varphi)[0,1)$ and every $n \in \mathbb{N}$, we have*

$$\|w_\alpha(f - M_n f)\| \le 2K_{w_\alpha}\left(f, \frac{\sqrt{2}}{n-1}\right).$$

–A Goodman-Sharma modification of MKZ operator

The Goodman-Sharma-type modification of MKZ operator (GS-MKZ) is given for natural n by

$$M_n^{[GS]} f(x) = \sum_{k=0}^{\infty} m_{n,k}(x) u_{n,k}(f),$$

$$u_{n,0}(f) = f(0), \quad u_{n,k}(f) = n \int_0^1 m_{n,k-1}(y) f(y) \frac{dy}{(1-y)^2}, \quad for \quad k \geq 1$$

where f is a Lebesgue integrable in $(0,1)$ function with a finite limit $f(0)$ at 0.

Using the equalities

$$B_{n+1}^{[AT]}(\tilde{f}, \tilde{x}) = M_n^{[GS]}(f, x),$$

$$\phi(\tilde{x}) D^2 \left(\mu(\tilde{x}) \tilde{f}(\tilde{x})\right) = \varphi(x) D^2 f(x),$$

the relations between the weights with tildes and without tildes and the result in Proposition 3.2 we have

Proposition 3.9. *Let w_α be given by* (3.8) *with* $\alpha_0, \alpha_1 \in [-1, 0]$*. Then for every* $f \in C(w_\alpha)[0,1) + W^2(w_\alpha \varphi)[0,1)$ *and every* $n \in \mathbb{N}$, $n \geq 2$, *we have*

$$\|w_\alpha(f - M_n^{[GS]} f)\| \leq 2 K_{w_\alpha}\left(f, \frac{2\sqrt{2}}{n-1}\right).$$

A better direct result and a strong converse result of type A are obtained in [8] using different arguments:

Proposition 3.10. *Let w_α be given by* (3.8) *with* $\alpha_0, \alpha_1 \in [-1, 0]$*. Then for every* $f \in C(w_\alpha)[0,1) + W^2(w_\alpha \varphi)[0,1)$ *and every* $n \in \mathbb{N}$, $n \geq 4$, *we have*

$$\|w_\alpha(f - M_n^{[GS]} f)\| \leq 2 K_{w_\alpha}\left(\tilde{f}, \frac{1}{2n}\right) \leq 13.7 \|w(f - M_n^{[GS]} f)\|_{[0,1)}.$$

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